

Fall 2016
OSE Qualifying Examination
Classical Electrodynamics

Instructions: Solve any 3 of the 5 problems in the exam. All problems carry equal points. Also, you may replace the complex number i occurring in Problems 2 and 3 by $-j$ if you prefer the more conventional engineering notation.

Possibly Useful Formulas

- Relation of spherical coordinates, (r, θ, ϕ) , to Cartesian coordinates:

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta.$$

Unit vectors:

$$\hat{r} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z};$$
$$\hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y}; \quad \hat{\theta} = \hat{\phi} \times \hat{r}.$$

- Laplacian in spherical coordinates:

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}.$$

- Azimuthally symmetric (ϕ -independent) solution of the Laplace equation in spherical polar coordinates:

$$V(r, \theta) = \sum_{\ell=0}^{\infty} \left(A_{\ell} r^{\ell} + \frac{B_{\ell}}{r^{\ell+1}} \right) P_{\ell}(\cos \theta),$$

where the first few Legendre polynomials are defined as

$$P_0(\cos \theta) = 1; \quad P_1(\cos \theta) = \cos \theta; \quad P_2(\cos \theta) = \frac{1}{2}(3 \cos^2 \theta - 1); \quad \text{etc.}$$

- Electric potential at position \vec{r} due to a point electric dipole of moment $p\hat{z}$ located at the origin:

$$V(\vec{r}) = \frac{\vec{p} \cdot \vec{r}}{4\pi\epsilon_0 r^3}.$$

- Polarization induced in a dielectric sphere of dielectric permittivity, ϵ , by a uniform external field, \vec{E} :

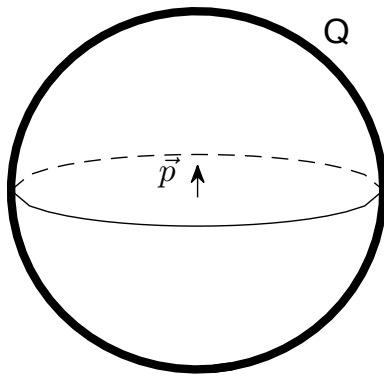
$$\vec{P} = 3\epsilon_0 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2} \right) \vec{E}, \quad \epsilon_r \equiv \frac{\epsilon}{\epsilon_0}.$$

- Force on an electric dipole due to an external electric field: $\vec{F} = (\vec{p} \cdot \nabla) \vec{E}$.
- Force on a volume current distribution: $\vec{F} = \int \vec{J} \times \vec{B} d\tau$.
- Fresnel formulas for the amplitude reflection coefficient of a plane wave incident at a planar interface between two dielectrics:

$$r_{\perp} = \frac{n \cos \theta - n' \cos \theta'}{n \cos \theta + n' \cos \theta'}; \quad r_{\parallel} = \frac{n' \cos \theta - n \cos \theta'}{n' \cos \theta + n \cos \theta'},$$

where \perp, \parallel refer, respectively, to polarizations perpendicular and parallel to the plane of incidence. The angles of incidence and refraction are θ and θ' , and n, n' are the refractive indices of the medium of incidence and the medium of transmission, respectively.

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1. Consider a point electric dipole of moment \vec{p} sitting at the center of a hollow conducting sphere of radius R and carrying a net charge of amount Q . The sphere sits on an insulating stand. By a proper choice of the coordinate system and in terms of the solution of the Laplace equation in spherical coordinates, involving the Legendre polynomials:

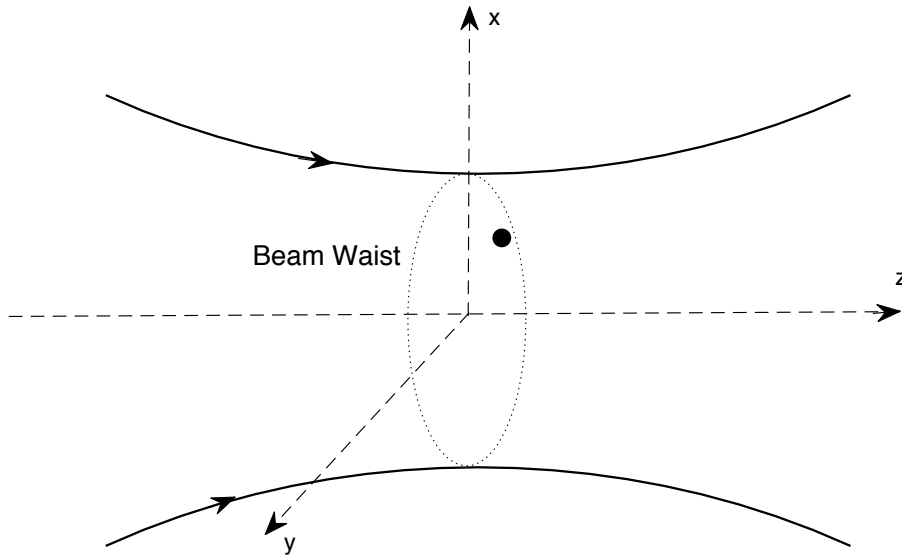


- (a) Calculate the potential everywhere inside the sphere.
- (b) Calculate the potential everywhere outside the sphere, including its surface.
- (c) Calculate the surface charge density on the inner surface of the sphere.
- (d) Argue, without detailed calculation, why the total induced charge on the inner surface must vanish.

2. A small dielectric sphere of linear dielectric permittivity ϵ and radius a that is much smaller than the wavelength, λ , of a monochromatic Gaussian optical beam resides in the plane of its sharpest focus. Let the beam width, w , be large compared to the wavelength so the beam may be adequately described as a transversely polarized beam. Let the beam be right circularly polarized. In its plane of sharpest focus, which we take to be the xy plane, the electric field of the beam traveling along the z axis has the following complex form:

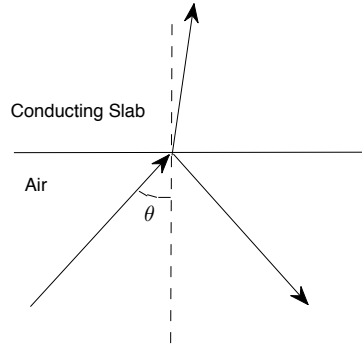
$$\vec{E}(\vec{\rho}, z = 0, t) = (\hat{x} + i\hat{y})E_0 \exp(-\rho^2/w^2) \exp(-i\omega t),$$

where $\vec{\rho} = (x\hat{x} + y\hat{y})$ is the transverse position vector in the xy plane and the origin is taken to be at the center of the beam.



- (a) If the sphere is placed off-beam-center at the point $(x_0, y_0, 0)$ in this plane, what time dependent polarization density will be induced in it in the small-radius approximation? (*Hint:* You can use, without deriving, the electrostatics of a dielectric sphere in a uniform electric field provided in the list of formulas to answer this question. Why is this adequate to calculate the spatial distribution of dielectric charges here?)
- (b) What time-averaged electrical force does the beam apply on the sphere? Will it attract the sphere toward the beam center or repel it away from the beam center?
- (c) Show that the time averaged electrical force on the sphere is radially inward. Due to this force, the sphere will perform simple harmonic oscillations in this plane, if released from rest from an off-axis position. If its mass is m , determine the frequency of oscillation.

3. A plane electromagnetic wave of angular frequency ω is incident at angle θ with respect to the normal of the surface of a semi-infinite slab occupying the region $z \geq 0$. Assume that the slab is highly conducting but non-magnetic, with a dielectric permittivity ϵ and conductivity σ that is large but not infinite, as shown in the figure.



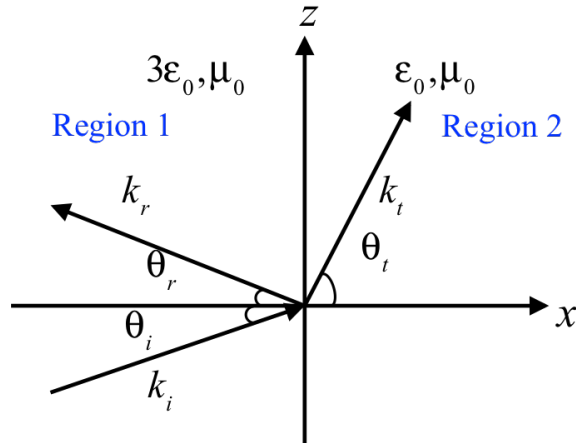
- (a) Show from Maxwell-Ampere's law that the slab may be regarded as a dielectric with a complex effective permittivity equal to $\epsilon + i\sigma/\omega \approx i\sigma/\omega$.
- (b) Show that the wave transmitted into the conductor is an evanescent plane wave. What is the characteristic depth, in terms of σ , ω , and certain electromagnetic constants, to which the transmitted field propagates inside the slab in the high-conductivity limit, $\sigma \gg \omega\epsilon$?
- (c) Show that in the high conductivity limit, the transmitted wave propagates essentially normally to the surface, regardless of the angle of incidence. Express n in terms of σ , ω and electromagnetic constants.

Surface Resistance [10 points]

What is the incident power density and the power absorbed per unit area in a sheet of brass ($\sigma = 1.5 \times 10^7$ mho/m) onto which a uniform plane wave is incident with a peak electric field of 1.0 V/cm at 10.0 GHz. (Hint, the surface resistance is given by $R_s = \frac{1}{\sigma\delta}$ where σ is the conductivity and δ is the skin depth.)

Question #5**Qualifying Exam, Fall 2016****Reflection and transmission [10 points]**

Consider a plane wave incident from a dielectric region with permittivity $\epsilon = 3\epsilon_0$ upon a half space with $\epsilon = \epsilon_0$ as shown in the figure below.



a) Find the Brewster angle for Region 1 [2 points].

b) Suppose the transmitted electric field is given by

$$\mathbf{E}_t = \hat{\mathbf{y}} \frac{E_0}{\sqrt{2}} \cos(k_{tx}x + k_{tz}z - \omega t) + E_0 \frac{\hat{\mathbf{z}} - \hat{\mathbf{x}}\sqrt{3}}{2\sqrt{2}} \sin(k_{tx}x + k_{tz}z - \omega t)$$

- i) Determine the incident and transmitted angles, θ_i and θ_t [2 points]
- ii) What is the polarization of the transmitted field? Be sure to specify the handedness (left or right) if necessary [2 points]
- iii) What is the polarization of the reflected wave? Be sure to specify the handedness (left or right) if necessary [2 points]
- iv) Give an expression for the incident electric field, \mathbf{E}_i , and the reflected electric field \mathbf{E}_r [2 points]