EQUATION SHEET

ELECTROMAGNETISM

Possibly Useful Formulas

- Relation of spherical coordinates, (r, θ, ϕ), to Cartesian coordinates:

$$x = r\,\sin\theta\,\cos\phi, \quad y = r\,\sin\theta\,\sin\phi, \quad z = r\,\cos\theta.$$

  Unit vectors:

$$\hat{r} = \sin\theta\,\cos\phi\,\hat{x} + \sin\theta\,\sin\phi\,\hat{y} + \cos\theta\,\hat{z};$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}; \quad \hat{\theta} = \hat{\phi} \times \hat{r}.$$

- Electric potential at position $\vec{r}$ due to a point electric dipole of moment $p\hat{z}$ located at the origin:

$$V(\vec{r}) = \frac{\vec{p}\cdot\vec{r}}{4\pi\epsilon_0 r^3}.$$

- Polarization induced in a dielectric sphere of dielectric permittivity, ϵ, by a uniform external field, $\vec{E}$:

$$\vec{P} = 3\epsilon_0\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)\vec{E}, \quad \epsilon_r \equiv \frac{\epsilon}{\epsilon_0}.$$

- Time-averaged power radiated by an oscillating electric dipole:

$$P = \frac{\mu_0 |p|^2 \omega^4}{12\pi c}.$$

- Time-averaged power radiated by an oscillating magnetic dipole:

$$P = \frac{\mu_0 |m|^2 \omega^4}{12\pi c^3}.$$

- Electric and magnetic fields at point $\vec{r}$ due to an oscillating electric dipole of moment $\vec{p}\exp(-i\omega t)$ at the origin -

$$\vec{E} = \frac{e^{i(kr-\omega t)}}{4\pi\epsilon_0}\left\{k^2\frac{(\hat{r}\times\vec{p})\times\hat{r}}{r} + \left(\frac{1}{r^3} - \frac{ik}{r^2}\right)[3(\hat{r}\cdot\vec{p})\hat{r} - \vec{p})]\right\};$$

$$\vec{B} = \frac{k^2 e^{i(kr-\omega t)}}{4\pi c\epsilon_0}(\hat{r}\times\vec{p})\left(\frac{1}{r} - \frac{ik}{r^2}\right); \quad k = \omega/c; \ \hat{r} = \frac{\vec{r}}{r}.$$

- Fresnel formulas for the amplitude reflection coefficient of a plane wave incident at a planar interface between two dielectrics:

$$r_\perp = \frac{n\cos\theta - n'\cos\theta'}{n\cos\theta + n'\cos\theta'}; \quad r_\parallel = \frac{n'\cos\theta - n\cos\theta'}{n'\cos\theta + n\cos\theta'},$$

  where $\perp, \parallel$ refer, respectively, to polarizations perpendicular and parallel to the plane of incidence. The angles of incidence and refraction are θ and θ', and n, n' are the refractive indices of the medium of incidence and the medium of transmission, respectively.

EQUATION SHEET

OPTICS

>> **Ray tracing matrices:**

Thin lens: $\begin{bmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{bmatrix}$ Free space: $\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$ Curved interface: $\begin{bmatrix} 1 & 0 \\ \frac{n_1 - n_2}{R n_2} 0 & \frac{n_1}{n_2} \end{bmatrix}$

>> **Snell's law:** $n_i Sin\theta_i = n_t Sin\theta_t$ **Lens-makers's formula:** $\frac{1}{f} = (\frac{n_2 - n_1}{n_1})(\frac{1}{R_1} - \frac{1}{R_2})$

> **Grating equation:** $Sin\,\theta' - Sin\,\theta = m\frac{\lambda}{d}$

>> **Gaussian beam:** $\frac{1}{q} = \frac{1}{R} - \frac{i\lambda}{n\pi w^2} = \frac{1}{R} - \frac{i}{\rho}$, $\rho = \rho_0\left(1 + \frac{z^2}{\rho_0^2}\right)$, $R = z\left(1 + \frac{\rho_0^2}{z^2}\right)$, $\rho_0 = \frac{\pi n w_0^2}{\lambda}$

$$\frac{E(x,y,z)}{E_{m,p}} = H_m\left[\frac{\sqrt{2}\,x}{w(z)}\right] H_p\left[\frac{\sqrt{2}\,y}{w(z)}\right] \times \frac{w_0}{w(z)} \exp\left[-\frac{x^2+y^2}{w^2(z)}\right] \times \exp\left\{-j\left[kz - (1+m+p)\tan^{-1}\left(\frac{z}{z_0}\right)\right]\right\} \times \exp\left[-j\frac{kr^2}{2R(z)}\right]$$

>> **Gaussian beam propagation:** $q_2 = \frac{Aq_1 + B}{Cq_2 + D}$

>> **Fabry-Perot:** Transmisison $T = \frac{(1-R)e^{-i\vec{k}.\vec{d}} + B}{1 - Re^{-i\delta}}$, $\delta = 2\varphi_r - 2\,\vec{k}.\vec{d}$, $|T|^2 = \frac{1}{1 + \frac{4R}{(1-R)^2 sin^2(\frac{\delta}{2})}}$

$$\delta = \sum gain/losses = 2\alpha L + ln\left(\frac{1}{R_1 R_2}\right) ,$$

Finess: $F = \frac{\Delta\nu_{FSR}}{\Delta\nu_{1/2}} = \frac{\pi\sqrt{R}}{1-R}$, Photon life time: $\tau = \frac{\tau_r}{\delta_c} = \frac{Q}{\omega_0} = \frac{\tau_{RT}}{1-S} = \frac{2nd/c}{1 - R_1 R_2}$ *Free spectral range*: $\Delta\nu_{FSR} = \frac{c}{2nd}$

>> **Fringe visibility:** $V = \frac{2|\gamma(\tau)|\sqrt{I_1 I_2}}{I_1 + I_2}$ $\gamma(\tau)$: complex degree of coherence

>> **Faraday cell rotation:** $\beta = VBd$, **Kerr effect:** $\Delta n = KE^2\lambda$, **Pokels effect:** $\Delta n = n^3\frac{r}{2}E$

>> **Irradiance:** $I = \langle S \rangle = \frac{1}{2\times\sqrt{\mu_0/\epsilon}} E_0^2 = \frac{nc\varepsilon_0}{2} E_0^2$

>> **Fresnel equations:**

$$r_{\parallel} = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} = \frac{n_t\cos\theta_i - n_i\cos\theta_t}{n_i\cos\theta_t + n_t\cos\theta_i} \qquad r_{\perp} = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)} = \frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t}$$

$$t_{\parallel} = \frac{2\sin\theta_i\cos\theta_i}{\sin(\theta_i + \theta_t)\cos(\theta_i - \theta_t)} = \frac{2n_i\cos\theta_i}{n_i\cos\theta_t + n_t\cos\theta_i} \qquad t_{\perp} = \frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_i + \theta_t)} = \frac{2n_i\cos\theta_i}{n_i\cos\theta_i + n_t\cos\theta_t}$$

>> **Fresnel integral:** z=distance between aperture plane (ξ,η) and observation plane (x,y)

$$U_p(x,y) = \frac{ie^{-ikz}}{\lambda z} e^{(-ik/2z)(x^2+y^2)} \iint_{-\infty}^{+\infty} U_A(\xi,\eta) \times e^{(-ik/2z)(\xi^2+\eta^2)} \times e^{-i(2\pi/\lambda z)(x\xi + y\eta)} d\xi d\eta$$

>> **Phase transformation of a spherical lens:** $t_l(x,y) = exp\left[\frac{ik}{2f}(x^2 + y^2)\right]$

>> **Fourier Transforms:** $\mathcal{F}\{g(x)\} = \int_{-\infty}^{\infty} g(x)e^{i2\pi(xf_x)}dx$, $\mathcal{F}^{-1}\{G(f_x)\} = \int_{-\infty}^{\infty} G(f_x)e^{-i2\pi(xf_x)}df_x$, $f_x = \frac{x}{\lambda z}$
(between space and spatial frequeny domain)

>> **Free space propagation** (*frequency domain*): $\mathcal{F}\{U(x)\} = \mathcal{F}\{U(\xi)\} \times exp(i\frac{2\pi^2}{k} z f_x)$

EQUATION SHEET

LASER

Einstein coefficients: $\frac{A_{21}}{B_{21}} = \frac{8\pi n^2 n_g h\nu^3}{c^3}$, $g_2 B_{21} = g_1 B_{12}$, $\frac{1}{\tau_{21,rad}} = A_{21}$

Natural linewidth : $\Delta\nu_n = \frac{1}{2\pi}\left[A_1 + A_2\right]$ where $A_2 = \sum_{j<2} A_{2j}$

Stimulated emission cross section: $\sigma(\nu) = A_{21}\frac{\lambda^2}{8\pi n^2} g(\nu)$

Gain (loss) for a two-level system: $\gamma(\nu) = \sigma(\nu)\left[N_2 - \frac{g_2}{g_1} N_1\right]$

Gain/Absorption: $\frac{dI_\nu}{dz} = \frac{\gamma_0 I_\nu}{1+\bar{g}(\nu)\left(I/I_s\right)}$

Saturation intensity: $I_s = \frac{h\nu}{\sigma(\nu)\tau_2}$