

EQUATION SHEET

ELECTROMAGNETISM

Possibly Useful Formulas

- Relation of spherical coordinates, (r, θ, ϕ) , to Cartesian coordinates:

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta.$$

Unit vectors:

$$\hat{r} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z};$$

$$\hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y}; \quad \hat{\theta} = \hat{\phi} \times \hat{r}.$$

- Electric potential at position \vec{r} due to a point electric dipole of moment $p\hat{z}$ located at the origin:

$$V(\vec{r}) = \frac{\vec{p} \cdot \vec{r}}{4\pi\epsilon_0 r^3}.$$

- Polarization induced in a dielectric sphere of dielectric permittivity, ϵ , by a uniform external field, \vec{E} :

$$\vec{P} = 3\epsilon_0 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2} \right) \vec{E}, \quad \epsilon_r \equiv \frac{\epsilon}{\epsilon_0}.$$

- Time-averaged power radiated by an oscillating electric dipole:

$$P = \frac{\mu_0 |p|^2 \omega^4}{12\pi c}.$$

- Time-averaged power radiated by an oscillating magnetic dipole:

$$P = \frac{\mu_0 |m|^2 \omega^4}{12\pi c^3}.$$

- Electric and magnetic fields at point \vec{r} due to an oscillating electric dipole of moment $\vec{p} \exp(-i\omega t)$ at the origin -

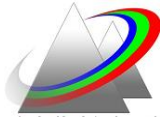
$$\vec{E} = \frac{e^{i(kr - \omega t)}}{4\pi\epsilon_0} \left\{ k^2 \frac{(\hat{r} \times \vec{p}) \times \hat{r}}{r} + \left(\frac{1}{r^3} - \frac{ik}{r^2} \right) [3(\hat{r} \cdot \vec{p})\hat{r} - \vec{p}] \right\};$$

$$\vec{B} = \frac{k^2 e^{i(kr - \omega t)}}{4\pi c \epsilon_0} (\hat{r} \times \vec{p}) \left(\frac{1}{r} - \frac{ik}{r^2} \right); \quad k = \omega/c; \quad \hat{r} = \frac{\vec{r}}{r}.$$

- Fresnel formulas for the amplitude reflection coefficient of a plane wave incident at a planar interface between two dielectrics:

$$r_{\perp} = \frac{n \cos \theta - n' \cos \theta'}{n \cos \theta + n' \cos \theta'}; \quad r_{\parallel} = \frac{n' \cos \theta - n \cos \theta'}{n' \cos \theta + n \cos \theta'},$$

where \perp, \parallel refer, respectively, to polarizations perpendicular and parallel to the plane of incidence. The angles of incidence and refraction are θ and θ' , and n, n' are the refractive indices of the medium of incidence and the medium of transmission, respectively.



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OPTICS

>> Ray tracing matrices:

$$\text{Thin lens: } \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \quad \text{Free space: } \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \quad \text{Curved interface: } \begin{bmatrix} 1 & 0 \\ \frac{n_1 - n_2}{Rn_2} & \frac{n_1}{n_2} \end{bmatrix}$$

>> Snell's law: $n_i \sin \theta_i = n_t \sin \theta_t$ Lens-makers's formula: $\frac{1}{f} = (n_2 - n_1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$ > Grating equation: $\sin \theta' - \sin \theta = m \frac{\lambda}{d}$ >> Gaussian beam: $\frac{1}{q} = \frac{1}{R} - \frac{i\lambda}{\pi w^2} = \frac{1}{R} - \frac{i}{\rho}$, $\rho = \rho_0 \left(1 + \frac{z^2}{\rho_0^2} \right)$, $R = z \left(1 + \frac{\rho_0^2}{z^2} \right)$, $\rho_0 = \frac{\pi n w_0^2}{\lambda}$

$$\frac{E(x, y, z)}{E_{m,p}} = H_m \left[\frac{\sqrt{2} x}{w(z)} \right] H_p \left[\frac{\sqrt{2} y}{w(z)} \right] \times \frac{w_0}{w(z)} \exp \left[-\frac{x^2 + y^2}{w^2(z)} \right] \times \exp \left\{ -j \left[kz - (1 + m + p) \tan^{-1} \left(\frac{z}{z_0} \right) \right] \right\} \times \exp \left[-j \frac{kr^2}{2R(z)} \right]$$

>> Gaussian beam propagation: $q_2 = \frac{Aq_1 + B}{Cq_1 + D}$ >> Fabry-Perot: Transmissison $T = \frac{(1-R)e^{-i\vec{k}\cdot\vec{d}} + B}{1 - Re^{-i\delta}}$, $\delta = 2\varphi_r - 2\vec{k}\cdot\vec{d}$, $|T|^2 = \frac{1}{1 + \frac{4R}{(1-R)^2 \sin^2(\frac{\delta}{2})}}$

$$\delta = \sum \text{gain/losses} = 2\alpha L + \ln \left(\frac{1}{R_1 R_2} \right),$$

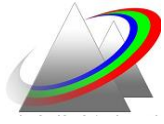
Finess: $F = \frac{\Delta\nu_{FSR}}{\Delta\nu_{1/2}} = \frac{\pi\sqrt{R}}{1-R}$, Photon life time: $\tau = \frac{\tau_r}{\delta_c} = \frac{Q}{\omega_0} = \frac{\tau_{RT}}{1-S} = \frac{2nd/c}{1-R_1 R_2}$ Free spectral range: $\Delta\nu_{FSR} = \frac{c}{2nd}$ >> Fringe visibility: $V = \frac{2|\gamma(\tau)|\sqrt{I_1 I_2}}{I_1 + I_2}$ $\gamma(\tau)$: complex degree of coherence>> Faraday cell rotation: $\beta = VBd$, Kerr effect: $\Delta n = KE^2 \lambda$, Pokels effect: $\Delta n = n^3 \frac{r}{2} E$ >> Irradiance: $I = \langle S \rangle = \frac{1}{2 \times \sqrt{\mu_0/\epsilon}} E_0^2 = \frac{nc\epsilon_0}{2} E_0^2$ >> Fresnel equations:

$$\begin{aligned} r_{\parallel} &= \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} = \frac{n_t \cos \theta_i - n_i \cos \theta_t}{n_i \cos \theta_t + n_t \cos \theta_i} & r_{\perp} &= -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)} = \frac{n_i \cos \theta_i - n_t \cos \theta_t}{n_i \cos \theta_i + n_t \cos \theta_t} \\ t_{\parallel} &= \frac{2 \sin \theta_i \cos \theta_t}{\sin(\theta_i + \theta_t) \cos(\theta_i - \theta_t)} = \frac{2n_i \cos \theta_i}{n_i \cos \theta_t + n_t \cos \theta_i} & t_{\perp} &= \frac{2 \sin \theta_t \cos \theta_i}{\sin(\theta_i + \theta_t)} = \frac{2n_i \cos \theta_i}{n_i \cos \theta_i + n_t \cos \theta_t} \end{aligned}$$

>> Fresnel integral: z=distance between aperture plane (ξ, η) and observation plane (x,y)

$$U_p(x, y) = \frac{ie^{-ikz}}{\lambda z} e^{(-ik/2z)(x^2+y^2)} \iint_{-\infty}^{+\infty} U_A(\xi, \eta) \times e^{(-ik/2z)(\xi^2+\eta^2)} \times e^{-i(2\pi/\lambda z)(x\xi+y\eta)} d\xi d\eta$$

>> Phase transformation of a spherical lens: $t_l(x, y) = \exp \left[\frac{ik}{2f} (x^2 + y^2) \right]$ >> Fourier Transforms: $\mathcal{F}\{g(x)\} = \int_{-\infty}^{\infty} g(x) e^{i2\pi(xf_x)} dx$, $\mathcal{F}^{-1}\{G(f_x)\} = \int_{-\infty}^{\infty} G(f_x) e^{-i2\pi(xf_x)} df_x$, $f_x = \frac{x}{\lambda z}$
(between space and spatial frequency domain)>> Free space propagation (frequency domain): $\mathcal{F}\{U(x)\} = \mathcal{F}\{U(\xi)\} \times \exp \left(i \frac{2\pi^2}{k} z f_x \right)$

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LASER

Einstein coefficients: $\frac{A_{21}}{B_{21}} = \frac{8\pi n^2 n_g h\nu^3}{c^3}$, $g_2 B_{21} = g_1 B_{12}$, $\frac{1}{\tau_{21,rad}} = A_{21}$

Natural linewidth: $\Delta\nu_n = \frac{1}{2\pi} [A_1 + A_2]$ where $A_2 = \sum_{j<2} A_{2j}$

Stimulated emission cross section: $\sigma(\nu) = A_{21} \frac{\lambda^2}{8\pi n^2} g(\nu)$

Gain (loss) for a two-level system: $\gamma(\nu) = \sigma(\nu) \left[N_2 - \frac{g_2}{g_1} N_1 \right]$

Gain/Absorption: $\frac{dI_\nu}{dz} = \frac{\gamma_0 I_\nu}{1 + \bar{g}(\nu)(I/I_s)}$

Saturation intensity: $I_s = \frac{h\nu}{\sigma(\nu)\tau_2}$