EQUATION SHEET

ELECTROMAGNETISM

Possibly Useful Formulas

- Relation of spherical coordinates, (r, θ, ϕ), to Cartesian coordinates:

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r \cos\theta.$$

  Unit vectors:

$$\hat{r} = \sin\theta \cos\phi \,\hat{x} + \sin\theta \sin\phi \,\hat{y} + \cos\theta \,\hat{z};$$
$$\hat{\phi} = -\sin\phi \,\hat{x} + \cos\phi \,\hat{y}; \quad \hat{\theta} = \hat{\phi} \times \hat{r}.$$

- Electric potential at position $\vec{r}$ due to a point electric dipole of moment $p\hat{z}$ located at the origin:

$$V(\vec{r}) = \frac{\vec{p}\cdot\vec{r}}{4\pi\epsilon_0 r^3}.$$

- Polarization induced in a dielectric sphere of dielectric permittivity, ϵ, by a uniform external field, $\vec{E}$:

$$\vec{P} = 3\epsilon_0 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right) \vec{E}, \quad \epsilon_r \equiv \frac{\epsilon}{\epsilon_0}.$$

- Time-averaged power radiated by an oscillating electric dipole:

$$P = \frac{\mu_0 |\vec{p}|^2 \omega^4}{12\pi c}.$$

- Time-averaged power radiated by an oscillating magnetic dipole:

$$P = \frac{\mu_0 |\vec{m}|^2 \omega^4}{12\pi c^3}.$$

- Electric and magnetic fields at point $\vec{r}$ due to an oscillating electric dipole of moment $\vec{p}\exp(-i\omega t)$ at the origin -

$$\vec{E} = \frac{e^{i(kr-\omega t)}}{4\pi\epsilon_0} \left\{ k^2 \frac{(\hat{r}\times\vec{p})\times\hat{r}}{r} + \left(\frac{1}{r^3} - \frac{ik}{r^2}\right) [3(\hat{r}\cdot\vec{p})\hat{r} - \vec{p}] \right\};$$
$$\vec{B} = \frac{k^2 e^{(kr-\omega t)}}{4\pi c\epsilon_0} (\hat{r}\times\vec{p}) \left(\frac{1}{r} + \frac{i}{kr^2}\right); \; k = \frac{\omega}{c}; \hat{r} = \frac{\vec{r}}{r}$$

- Fresnel formulas for the amplitude reflection coefficient of a plane wave incident at a planar interface between two dielectrics:

$$r_\perp = \frac{n\cos\theta - n'\cos\theta'}{n\cos\theta + n'\cos\theta'}; \quad r_\parallel = \frac{n'\cos\theta - n\cos\theta'}{n'\cos\theta + n\cos\theta'},$$

  where $\perp, \parallel$ refer, respectively, to polarizations perpendicular and parallel to the plane of incidence. The angles of incidence and refraction are θ and θ', and n, n' are the refractive indices of the medium of incidence and the medium of transmission, respectively.

OPTICS

Ray tracing matrices:

Thin lens: $\begin{bmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{bmatrix}$ Free space: $\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$ Curved interface: $\begin{bmatrix} 1 & 0 \\ \frac{n_1-n_2}{Rn_2} & \frac{n_1}{n_2} \end{bmatrix}$ (R: radius)

Snell's law: $\boldsymbol{n_i Sin\theta_i = n_t Sin\theta_t}$ **Lens-makers's formula:** $\frac{1}{f} = (\frac{n_2-n_1}{n_1})(\frac{1}{R_1} - \frac{1}{R_2})$ (R_i: radius)

Grating equation: $\boldsymbol{Sin\,\theta' - Sin\,\theta = m\frac{\lambda}{d}}$

Gaussian beam: $\frac{1}{q} = \frac{1}{R(z)} - \frac{j\lambda_0}{n\pi w^2(z)}$ $w(z) = w_0^2\left(1+\frac{z^2}{z_0^2}\right)$, $R(z) = z\left(1+\frac{z_0^2}{z^2}\right)$, $z_0 = \frac{\pi n w_0^2}{\lambda_0}$

$$\frac{E(x,y,z)}{E_{m,p}} = H_m\left[\frac{\sqrt{2}\,x}{w(z)}\right] H_p\left[\frac{\sqrt{2}\,y}{w(z)}\right] \times \frac{w_0}{w(z)} \exp\left[-\frac{x^2+y^2}{w^2(z)}\right] \times \exp\left\{-j\left[kz-(1+m+p)\tan^{-1}\left(\frac{z}{z_0}\right)\right]\right\} \times \exp\left[-j\frac{kr^2}{2R(z)}\right]$$

Cavity stability condition: $\boldsymbol{0 \le \frac{A+D+2}{4} \le 1}$

Gaussian beam propagation from point 1 to point 2 : $\boldsymbol{q_2 = \frac{Aq_1+B}{Cq_1+D}}$

Gaussian beam in a cavity: $\boldsymbol{\frac{1}{q} = -\frac{A-D}{2B} - j\frac{\sqrt{1-\left(\frac{A+D}{2}\right)^2}}{B}}$

Fabry-Perot *Transmisison and Reflection* (when $r_1=r_2=r$): $t = \frac{E_t}{E_i} = \frac{(1-r^2)e^{-i\delta/2}}{1-r^2e^{-i\delta}}$, $r = \frac{E_r}{E_i} = \frac{r^2(e^{-i\delta}-1)}{1-r^2e^{-i\delta}}$, $\delta = 2kd$

$$T = \left|\frac{E_t}{E_i}\right|^2 = \frac{(1-r^2)^2}{1+r^4-2r^2cos(\delta)} \quad and \quad R = \left|\frac{E_r}{E_i}\right|^2 = 1-T$$

$\boldsymbol{r}$ is *reflection coefficient* and $\boldsymbol{R}$ (= $\boldsymbol{r^2}$) is *reflectance*. For asymmetric case ($r_1 \neq r_2$) : $r^2 \rightarrow r_1 \times r_2$

Finess: $\boldsymbol{F = \frac{\Delta\nu_{FSR}}{2\Delta\nu_{1/2}} = \frac{\pi r}{1-r^2}}$, Photon life time: $\tau = \frac{\tau_r}{\delta_c} = \frac{Q}{\omega_0} = \frac{\tau_{RT}}{1-S} = \frac{2nd/c}{1-R^2}$, (*note that* $2\Delta\nu_{1/2}$ *is the FWHM*)

Free spectral range: $\Delta\nu_{FSR} = \frac{c}{2nd}$, Quality fator: $\boldsymbol{Q = \frac{\nu}{\nu_{FSR}} \times F = \frac{\nu}{2\Delta\nu_{1/2}}}$

Fringe visibility (two path inteference with single source): $\boldsymbol{V = \frac{2|\gamma(\tau)|\sqrt{I_1 I_2}}{I_1+I_2}}$, $\boldsymbol{\gamma(\tau)}$: complex degree of temporal coherence

Faraday cell rotation: $\boldsymbol{\beta = VBd}$, **Kerr effect:** $\boldsymbol{\Delta n = KE^2\lambda}$, **Pockels effect:** $\boldsymbol{\Delta n = n^3\left(\frac{r_{eo}}{2}\right)E}$

Irradiance: $\boldsymbol{I = \langle S\rangle} = \frac{1}{2\times\sqrt{\mu_0/\epsilon}}E_0^2 = \frac{nc\varepsilon_0}{2}E_0^2$

Fresnel equations:

$$r_\parallel = \frac{\tan(\theta_i-\theta_t)}{\tan(\theta_i+\theta_t)} = \frac{n_t\cos\theta_i - n_i\cos\theta_t}{n_i\cos\theta_t + n_t\cos\theta_i} \qquad r_\perp = -\frac{\sin(\theta_i-\theta_t)}{\sin(\theta_i+\theta_t)} = \frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t}$$

$$t_\parallel = \frac{2\sin\theta_i\cos\theta_i}{\sin(\theta_i+\theta_t)\cos(\theta_i-\theta_t)} = \frac{2n_i\cos\theta_i}{n_i\cos\theta_t + n_t\cos\theta_i} \qquad t_\perp = \frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_i+\theta_t)} = \frac{2n_i\cos\theta_i}{n_i\cos\theta_i + n_t\cos\theta_t}$$

OPTICS (continued)

Fresnel integral: z=distance between aperture plane (ξ,η) and observation plane (x,y):

$$U_p(x,y) = \frac{ie^{-ikz}}{\lambda z} e^{(-ik/2z)(x^2+y^2)} \iint_{-\infty}^{+\infty} U_A(\xi,\eta) \times e^{(-ik/2z)(\xi^2+\eta^2)} \times e^{-i(2\pi/\lambda z)(x\xi+y\eta)} d\xi d\eta$$

Phase transformation of a spherical lens: $t_l(x,y) = exp\left[\frac{ik}{2f}(x^2+y^2)\right]$

Fourier Transforms: $\mathcal{F}\{g(x)\} = \int_{-\infty}^{\infty} g(x)e^{i2\pi(xf_x)}dx$, $\mathcal{F}^{-1}\{G(f_x)\} = \int_{-\infty}^{\infty} G(f_x)e^{-i2\pi(xf_x)}df_x$, $f_x = \frac{x}{\lambda z}$

(between space and spatial frequeny domain)

Free space propagation (*frequency domain*): $\mathcal{F}\{U(x)\} = \mathcal{F}\{U(\xi)\} \times exp(i\frac{2\pi^2}{k}zf_x)$

Jones matrices

Linear polarizer : *TA horizontal:* $\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$, *TA vertical* $\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$

Phase retarders : $\begin{bmatrix}e^{i\delta_x} & 0\\0 & e^{i\delta_y}\end{bmatrix}$, *Rotator* : $\begin{bmatrix}\cos\beta & -\sin\beta\\\sin\beta & \cos\beta\end{bmatrix}$

Doppler shift when $v << c$: $\frac{\lambda'}{\lambda} \approx 1 - \frac{v}{c}$

Radiation pressure (when a wave is completely absorbed): $P = \frac{\langle S\rangle}{c} = \frac{I_{inc}}{c}$

Drude model

Complex permittivity: $\tilde{\varepsilon}(\omega) = \varepsilon'(\omega) + j\varepsilon''(\omega)$

Complex refractive index: $\tilde{n}(\omega) = n(\omega) + j\kappa(\omega)$

$$n = \sqrt{\frac{1}{2}\left[\varepsilon' + \sqrt{\varepsilon'^2 + \varepsilon''^2}\right]}$$

$$\kappa = \sqrt{\frac{1}{2}\left[-\varepsilon' + \sqrt{\varepsilon'^2 + \varepsilon''^2}\right]}$$

Plasma frequency: $\omega_P^2 = \frac{Ne^2}{\varepsilon_0 m_0}$

$$\varepsilon'(\omega) = n^2(\omega) - \kappa^2(\omega) = 1 - \frac{\omega_P^2\tau^2}{\omega^2\tau^2+1}$$

$$\varepsilon''(\omega) = 2\,n(\omega)\,\kappa(\omega) = \frac{1}{\omega}\frac{\omega_P^2\tau}{\omega^2\tau^2+1}$$

Optics of metal surfaces

Refractive index of a dielectric n_1 is real. Complex refractive index of metals: $\widetilde{n_2} = n_2 + j\kappa_2$.

Complex reflection coefficient at normal incidence: $r = \frac{E_r}{E_i} = \rho e^{j\varphi} = \frac{n_1-(n_2+j\kappa_2)}{n_1+(n_2+j\kappa_2)}$

where:

$$\rho^2 = \frac{(n_1-n_2)^2+\kappa_2^2}{(n_1+n_2)^2+\kappa_2^2}; \quad \tan\varphi = \frac{2\kappa_2 n_1}{n_2^2+\kappa_2^2-n_1^2}$$

OPTICS (continued)

Electro-optic effects

Phase shift between two orthogonal components of linearly-polarized electric field for transverse Pockels effect:

$$\theta_T = \frac{2\pi}{\lambda} L\,\Delta n = \frac{\pi}{\lambda} L n_0^3 r \frac{U}{d}$$

Temporal coherence

Flux density $S(\tau)$ of interference pattern and coherence (normalized autocorrelation) function $\gamma(\tau)$:

$$S(\tau) = S_0[1 + \gamma(\tau)]$$

Coherence function $\gamma(\tau)$ and normalized spectral density function $P(\omega)$:

$$\gamma(\tau) = \int_0^\infty P(\omega)\cos(\omega\tau)\,d\omega$$

Normalized spectral density function $P(\omega)$ and coherence function $\gamma(\tau)$:

$$P(\omega) = 2\int_{-\infty}^{\infty} \gamma(\tau)\ \cos(\omega\,\tau)\ d\tau = 4\int_0^{\infty} \gamma(\tau)\ \cos(\omega\,\tau)\ d\tau$$

Definition of time averaging:

$$\langle g\rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} g[E(t)]dt$$

-- LASER --

Einstein coefficients: $\frac{A_{21}}{B_{21}} = \frac{8\pi n^3 h\nu^3}{c^3}$, $g_2 B_{21} = g_1 B_{12}$, $\frac{1}{\tau_{21,rad}} = A_{21}$

Natural linewidth : $g(\nu) = \frac{\Delta\nu}{2\pi\left[(\nu_0-\nu)^2+\left(\frac{\Delta\nu}{2}\right)^2\right]}$, $\Delta\nu = \frac{1}{2\pi}\left(\frac{1}{\tau_2}+\frac{1}{\tau_1}\right) = \frac{1}{2\pi}(A_1 + A_2)$, $A_2 = \sum_{j<2} A_{2j}$

Stimulated emission cross section: $\sigma(\nu) = A_{21}\frac{\lambda^2}{8\pi n^2} g(\nu)$

Gain (loss) coeeficient for a two-level system: $\gamma(\nu) = \sigma(\nu)\left[N_2 - \frac{g_2}{g_1}N_1\right]$

Gain/Absorption: $\frac{dI_\nu}{dz} = \frac{\gamma_0 I_\nu}{1+\bar{g}(\nu)\left(I_\nu/I_s\right)}$, $\ln\left(\frac{I_{out}}{G_0 I_{in}}\right) = \frac{I_{in} - I_{out}}{I_s}$ where $G_0 = e^{\gamma_0 l_g}$

Saturation intensity: $I_s = \frac{h\nu}{\sigma(\nu)\tau_2}$

-- CONSTANTS --

$h=6.62\times10^{-34}$ J.s; $c=3\times10^{8}$ m/s; $e=1.6\times10^{-19}$ C; $k=1.38\times10^{-23}$ J.K^{-1}; $\varepsilon_0=1.85\times10^{-12}$ F/m; $m_0 = 9.109 \times 10^{-31}$ kg.