Comprehensive Examination - Optics

**Thursday, August 18, 2005, 9:00 am – 12 noon**

**Part I: Advanced Optics I & II**

Solve 4 out of the 6 problems. Each one carries the same number of points as any other.

## Problem I.1

A y-polarized plane wave of wavelength $\lambda$ is propagating in the $+z$ direction (cf. Fig. 1). The plane wave is incident on a thin phase screen at $z = z_0$ with the transmission function:

$$T(x) = e^{i(\frac{2\pi}{L}x)},$$

(a) (1 pt.) Write down the amplitude and phase of the transmitted electric field at $z = z_0^+$ (i.e. just right of the screen)?

(b) (4 pts) Find the angular diffraction pattern of this phase screen at infinity.

(c) (3 pts) If $L = 2\lambda$ show that the amplitude of the electric field in the far field is proportional to:

$$\cos\frac{k\theta}{2}\cos\frac{k\theta}{4}\text{sinc}\left(\frac{k\theta}{8} + \pi\right)$$

where

$$\theta \approx \sin\theta = \frac{k_x}{k}$$

(d) (1 pt.) Sketch the intensity transmission as a function of $\theta$.

(e) (1 pt.) Why is there only one diffraction order?

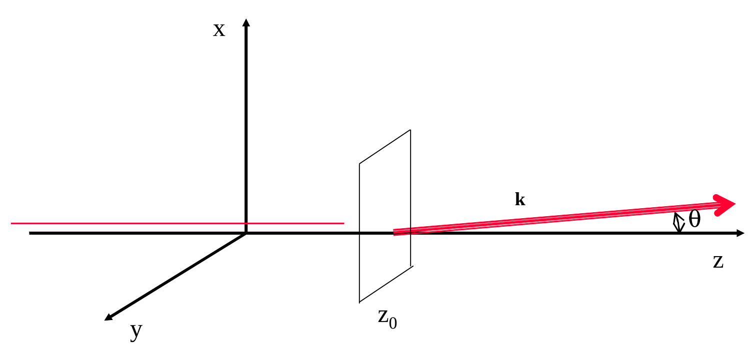


Figure 1: System of coordinates relative to the phase plate

## Problem I.2

Consider the bi-convex thick lens shown in Fig. 2 constructed from a material with a refractive index $n$ and thickness $L$.

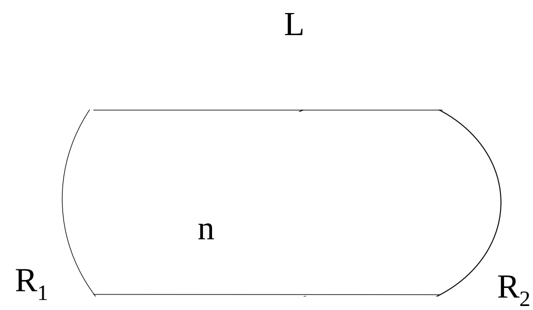


Figure 2: Thick biconvex lens.

(a) (4 pts) Derive the transfer ray matrix of the system.

(b) (2 pts) Given the radii of curvature, obtain the thickness for which the lens will act as a telescope and write the new ray matrix.

(c) (2 pts) Find the condition for the magnitude of the transverse magnification to be 2. What is the angular magnification in this case (this question refers to the system when used as a telescope).

(d) (2 pts) A collimated laser beam with a diameter $d$ is incident on the telescope in part (c). Draw the ray diagram through the system and identify the focus and the exiting beam diameter and angle.

**Problem I.3.**

A Gaussian beam having a beam waist $w_{01}$ and a Rayleigh range $z_{01}$ is incident (at its waist) on a thin lens with focal length $f$, as shown in Fig. 3. The beam is to be focused on the back-surface of a dielectric slab of thickness $d$ and refractive index $n$.

(a) (2 pts) Obtain the ABCD matrix for propagation from plane $P_1$ to plane $P_2$.

(b) (3 pts) Derive the distance $L$ between the lens and dielectric slab.

(c) (3 pts) What is the new beam waist? (Does the beam waist depend on $n$?)

(d) (2 pts) What is the distance $\Delta L$ by which we should move the slab in order to focus the beam on the front surface? Is the beam waist, in this case, different from part (c)?

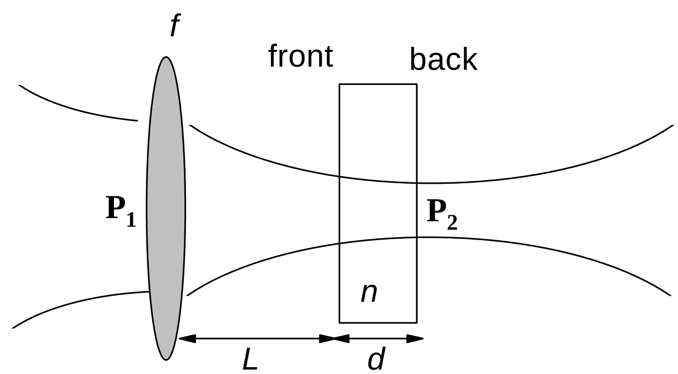


Figure 3: Lens problem.

## Problem I.4

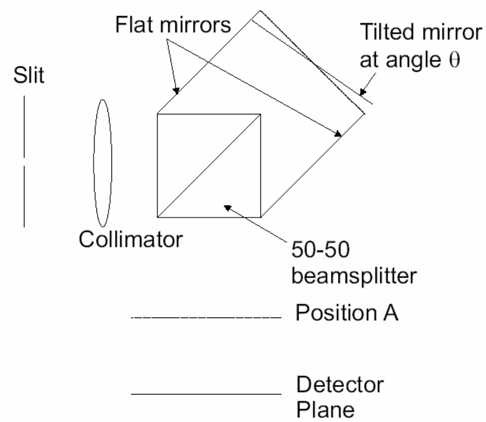


Figure 4: Shear interferometer. The angle $\theta$ of the tilted mirror is small.

A different type of interferometer is the shear interferometer depicted in Fig. 4.

- (4 pts) Sketch the ray paths through the interferometer to the detector plane. Neglect beam divergence and assume that the detector plane is located so as to maximize overlap between the two emerging beams.
- (4 pts) Assume that the radiation impinging on the slit has a flat spectral profile between the wavelengths $\lambda_1$ and $\lambda_2$. What does the interference pattern look like at the detector plane?
- (2 pts) What is the effect of putting a cylinder lens with its axis of curvature *in the plane of the figure* at position A, one focal length away from the detector plane?

**Problem I.5.**

A transmission grating having a field transmission function $\tau(x) = \cos(\pi x/\Lambda)^2$ ($\Lambda \gg \lambda$ where $\lambda$ is the wavelength) is illuminated by plane waves and is imaged through a single thin lens ($\infty$ diameter) as shown in Fig. 5. This is a two-dimensional problem.

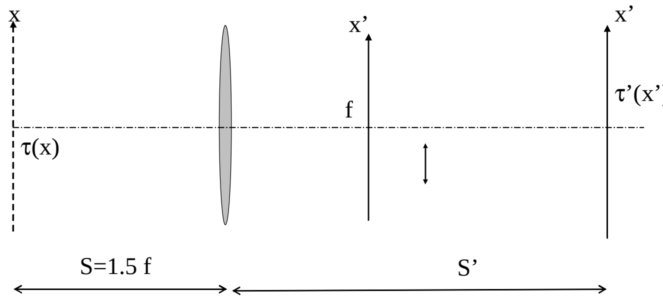


Figure 5: Transmission grating.

(a) (3 pt.) Derive the flux density distribution at the image plane and plot it in comparison with the transmission function

(b) (4 pts.) Derive and plot the field amplitude distribution at the focal plane

(c) (3 pts.) Derive and plot the flux density distribution at the image plane if focal plane filter-$B$ is used

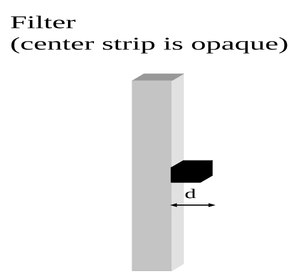


Figure 6: Side view of the filter. The height of the center strip, $\Delta x$ is much less than $(f\lambda/\Lambda)$.

## Problem I.6.

A plane wave of wavelength 600 nm illuminates a long, narrow slit (Fig. 7) of width $10^{-4}$ m producing a far-field diffraction pattern that is described by

$$S(u) = S(0)\mathrm{sinc}^2(2\pi u x_0),$$

where $2x_0$ is the slit width and $u = (\alpha - \alpha')/\lambda$.

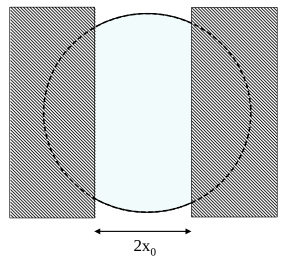


Figure 7: narrow slit

(a) (2 pts.) the angle at the slit from the center to the second minimum

(b) (1 pt) How does this angle change if the slit width is decreased 10 times?

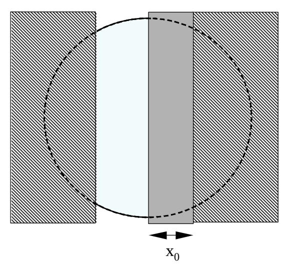


Figure 8: narrow slit with half of the opening covered with a neutral density filter.

(c) Half of the slit opening opening is covered with a neutral density filter that cuts the power through that half by a factor of 4. The filter has a width $x_0$ and splits the opening into two segments, as shown in Fig. ??.

  (a) (5 pts.) Determine the far field diffraction pattern

  (b) (2 pts.) Calculate the angle (in radians) at the slit between the first and second minimum on the side of the filter.

Comprehensive Examination - Optics

**Thursday, August 19, 2004, 2:00 pm – 5:00 pm**

**Part II: General part**

## 2.1 Section 1

**Answer the seven questions below, give *estimates* when appropriate, use the aid of a sketch, and no more than 10 sentences per question. ANSWER EACH QUESTION ON A SEPARATE PAGE. — 1 points each question**

1. What is the orientation of the linear polarizer in sunglasses? (Hint: Consider the most common geometry for generating glare).

2. You are given a pair of right-angle prisms. Describe how you would construct a variable transmission device. Can you comment on the wavelength selectivity of your device?

3. How does the "principle of causality" manifest itself in the optical properties of materials? Explain.

4. By making sensible estimates of the surface area of the human body and its radiant temperature, calculate its radiant emittance and the total radiant power emitted.

5. A laser emits 50 mW at a wavelength of 500 nm. If the beam is collimated to a diameter of 10 mm, estimate the maximum photo-current which could be generated if the beam irradiates uniformly a photodiode of diameter 1 mm.

6. You have two microscope objectives at your disposal. They are specified as follows: 100x, NA = 0.4 (objective A) and 40x, NA = 0.8 (objective B). Which one has the better resolution, why? Estimate the resolution (in microns) of the two objectives assuming illumination with 500-nm light. Explain a scenario where one would use the objective with the weaker resolution.

7. A Brewsters angle window is often used in a laser cavity to obtain polarization purity. Describe how the Brewsters angle window ensures a particular polarization state, and which state is it?

## 2.2 Answer all six following general problems (2 points each question)

### 2.2.1

A one-mW HeNe laser (632 nm) is directed toward the Moon. Estimate the power that a detector of 1 cm diameter on the Moon is able to collect. How many photons per second does this power correspond to? Neglect any scattering and absorption losses. Clearly state your assumptions. (Hint: distance Earth - Moon 400 000 km.)

### 2.2.2

Consider the experimental setup sketched in the figure. A laser emits a polarized beam of power $P_0$.

(a) A linear polarizer (P) is inserted between laser and detector and rotates with an angular velocity $\omega$ . Sketch the trace that one sees on an oscilloscope (OS). Make sure you calibrate both axes using the quantities given. Explain.

(b) The polarizer is at a fixed orientation in the beam path with its pass direction parallel to the polarization emitted by the laser. A half-wave plate (HWP) that rotates with angular frequency $\omega$ is inserted between laser and polarizer. Sketch the oscilloscope trace. Make sure you calibrate both axes in terms of $P_0$ and $\omega$. Explain.

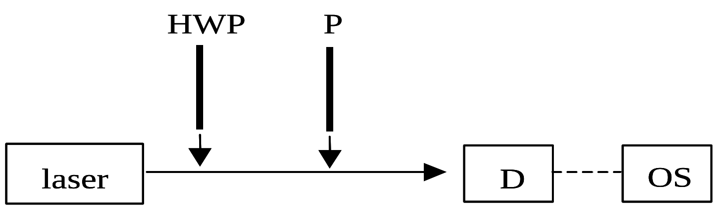


Figure 9: Filters.

### 2.2.3

(a) A recent mission sought to launch a solar-sail powered spacecraft, which would have been pushed away from the sun by photons reflecting off the metallized sail.

Suppose the radiation force compensates for the gravitational force, so that the spacecraft stays at the same distance from the sun as the earth, but not orbit the sun, remaining stationary. For a 1000 kg ship, how large would the sail have to be?

It is 1.5 $\times 10^{11}$ m to the sun from the earth. The gravitational force at this distance from the sun is 6 Newton.

The sun radiates 3.9 $\times 10^{26}$ W.

(b) If the solar-sail spaceship were closer to the sun, the solar intensity would be higher, of course. Would the ship then accelerate away from the sun?

### 2.2.4

A grating consists of alternating opaque and transparent bars, each of width 1 micron. It is illuminated by an expanded, essentially parallel laser beam, $\lambda = 500$ nm, at normal incidence. You wish to use a lens of focal length f = 20 mm to make a 10X magnified image of the grating on a screen.

(a) Where should the lens and the screen be placed?

(b) What is the minimum size (diameter) of the lens that will give an image?

(c) Describe the image obtained with this minimum diameter lens.

### 2.2.5

Two rectangular blocks are stacked as shown in Fig. 10; the top block has an index of 1.2, the bottom an index of 1.5.

(a) What range of incident angle $\alpha$ will cause Ray 1 to be totally internally reflected at the bottom block?

(a) What range of incident angle $\beta$ will cause Ray 2 to be totally internally reflected at the bottom block?

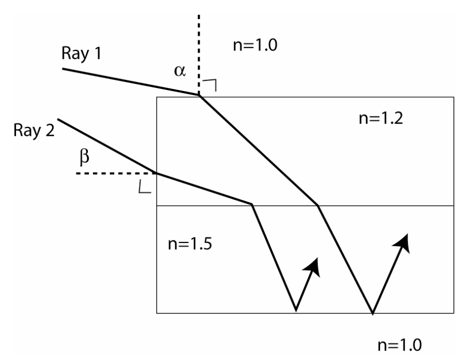


Figure 10: Stack of rectangular blocks.

### 2.3 Answer the following general problem (3 points)

#### 2.3.1

A laser at 500 nm provides a TEM00 Gaussian beam with a $1/e^2$ waist radius of $w_0 = 0.2$ mm. The waist is located at the output coupler.

(a) The beam power is 1 mW. What is the maximum intensity, just outside the output coupler?

(b) Suppose the lens has a focal length of 8000 mm. In what position(s) could you place this lens so that the beam after passing through it, has a plane wavefront?

(c) The lens has now a focal length of 80 mm. Could you place it anywhere, so that the beam after passing through it, has a plane wavefront?

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**Friday, August 20, 2004, 9:00 am – 12 noon**

**Part III: Electromagnetism**

Solve 4 out of the 6 problems. Each one carries the same number of points as any other.

## Problem III.1

An electromagnetic wave of frequency $\omega$ is incident on a region containing $10^{16}$ electrons/cc. The collision rate of the electrons is $10^{8}$ s$^{-1}$. Make appropriate approximations.

(a) (2 pts.) Write the equation of motion for the electron under the influence of the electromagnetic field.

(b) (3 pts.) Given that the field $\tilde{E} = \tilde{\mathcal{E}} \exp(i\omega t)$ is linearly polarized, driving an electron current, use Maxwell's equations to derive an expression for the dielectric constant of this electron gas.

(c) (2 pts.) What defines the absorption coefficient in this problem? Find the absorption length at a frequency of $2 \cdot 10^{8}$ s$^{-1}$.

(d) (1 pts.) What is the phase velocity at a frequency $\omega = 10^{11}$ s$^{-1}$?

(e) (2 pts.) Derive an expression for the group velocity, and find its value at $\omega = 10^{11}$ s$^{-1}$.

## Problem III.2

A thin insulating rod of length $2d$ carrying two equal point charges, each of amount $q$, is set into uniform circular motion in the $xy$ plane with angular frequency $\omega$, as sketched in Fig. 11. Assume that $d\,\omega/c << 2\pi$.

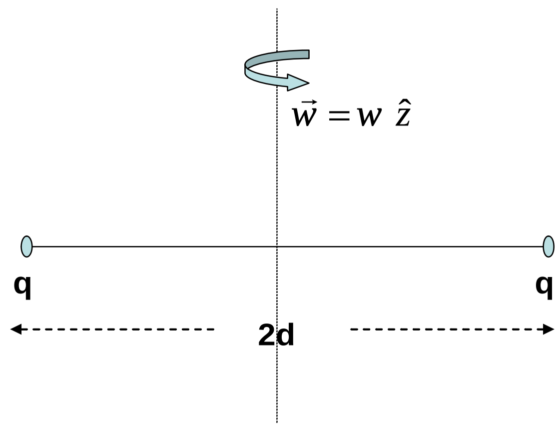



Figure 11: Spinning rod.

(a) If the rod rotates about its center, then argue why the charges can not radiate either electric or magnetic dipole radiation.

(2 pts)

(b) If the axis of rotation is shifted from the center of the rod, say by amount $\Delta < d$, then show that radiation is possible in the dipole order. Determine the total electric field of radiation in the far zone in the $xz$ plane, emitted at a polar angle $\theta$ relative to the $z$ axis. Comment on the nature of the field polarization for $\theta = 0, \pi/4$, and $\pi/2$.

(5 pts)

(c) Find the total time-averaged power radiated by the charges in the case (b).

(3 pts)

**Useful formulas:**

- Electric-dipole radiation in the far zone:

$$\vec{E} = \frac{k^2}{4\pi\epsilon_0}\frac{e^{ikr-i\omega t}}{r}(\hat{n}\times\vec{p})\times\hat{n}, \quad \vec{B} = \frac{k}{\omega}\hat{n}\times\vec{E};$$

- Magnetic-dipole radiation in the far zone:

$$\vec{B} = \frac{k^2\mu_0}{4\pi}\frac{e^{ikr-i\omega t}}{r}(\hat{n}\times\vec{m})\times\hat{n}, \quad \vec{E} = -\frac{c^2 k}{\omega}\hat{n}\times\vec{B}.$$

Here $\hat{n}$ is the radial unit vector along the observation direction; $\vec{p}, \vec{m}$ are the complex electric and magnetic dipole-moment coefficient vectors; and $k = \omega/c$. The other symbols have their usual meaning.

## Problem III.3

(a) (4 pts.) What is the physical thickness of a layer (antireflection layer) on a substrate with dielectric constant $\epsilon = 4\epsilon_0$, in order to have a minimum reflectivity for $f_0 = 20$ GHz (the incident beam originates from air and is normal to the interface). What is the dielectric constant of the layer?

(b) (3 pts.) Find the reflectivity at 23 and 17 GHz.

(c) (2 pts.) Qualitatively explain how this reflection will change when circularly polarized light is incident at $\theta_0 \neq 0$.

(d) (1 pt.) Qualitatively explain what happens if the dielectric slab is lossy.

### Problem III.4

A current, with current density given by $\underline{J} = \hat{x}(1\text{A/m}^2)e^{j(\omega t - \beta z)}$, is driven in a medium whose relative dielectric constant at the frequency $\omega$ is given in Cartesian coordinates by the tensor:

$$\epsilon_r = \begin{bmatrix} 1 & 1-j & 0 \\ 1+j & j & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) (2 pts.) Find the AC conductivity tensor at the frequency $\omega$.

(b) (2 pts.) Find the electric susceptibility tensor at the frequency $\omega$.

(c) (2 pts.) Find the electric field associated with this current.

(d) (2 pts.) Find the corresponding electric flux density, $\underline{D}$, and the polarization vector, $\underline{P}$.

(e) (2 pts.) What is the polarization of this wave?

## Problem III.5

A metal bar of mass $m$ slides without friction on two parallel conducting rails a distance $\ell$ apart (see Fig. 12). A resistor $R$ is connected across the rails and a uniform magnetic field, $\mathbf{B}$, pointing into the page, fills the entire region.

(a) (2 pts.) If the bar moves to the right with a velocity $v$, what is the current in the resistor? Please give the magnitude and direction of the current ($v \ll c$).

(b) (2 pts.) What is the magnetic force on the bar? Please give the magnitude and direction of the force.

(c) (2 pts.) If the bar is given an initial speed $v_0$ to the right at $t = 0$, what is its speed at any later time $t$?

(d) (2 pts.) Show that the total energy delivered to the resistor in the limit of $t \to \infty$ is $\frac{1}{2}mv_0^2$.

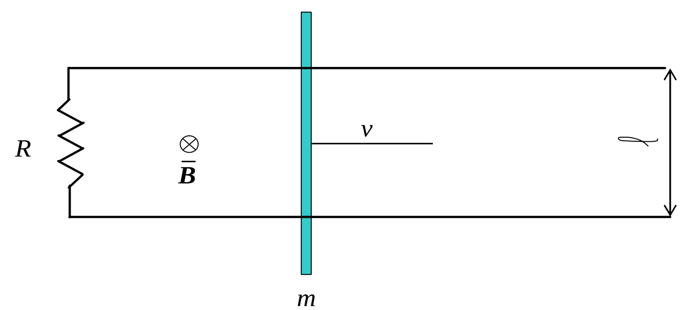



Figure 12: Sliding bar of mass $m$ closing a resistive rectangular circuit in a magnetic field.

### Problem III.6

A capacitor is formed by the small gap $w$ in a wire of radius $a$ (see Fig. 13), where $w \ll a$, and the initial charge/ area $\sigma$ is null on either side of the gap. At $t = 0$, a constant current $I$ is sent through the wire.

(a) (3 pts.) Derive the electric, $\mathbf{E}(s,t)$, and magnetic, $\mathbf{B}(s,t)$, fields as a function of time $t$ ($t \geq 0$) and distance $s$ from the wire axis.

(b) (4 pts.) Find the energy density $u_{em}(s,t)$ in electric and magnetic fields, and the Poynting vector $S(s,t)$ in magnitude and direction, in the gap.

(c) (3 pts.) Determine the total energy in the gap (i.e. out to radius $s = a$) as a function time. Calculate also the total power flowing into the gap. Confirm that the power input equals the rate of increase of energy in the gap with time. (If you are concerned about fringe fields, do this calculation for some radius $s = b < a$ well inside the gap.)

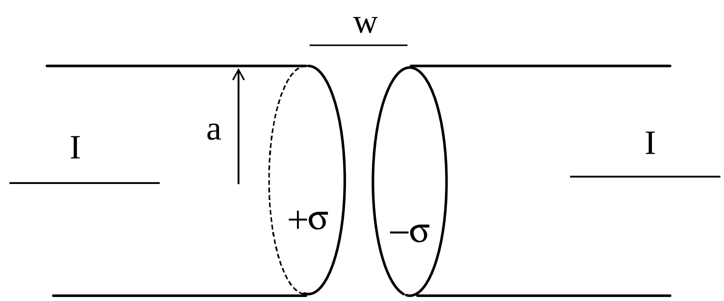



Figure 13: Interrupted wire.

Comprehensive Examination - Optics
**Friday, August 19, 2005, 2:00 pm – 5:00 pm**
**Part IV: Laser Physics**
Solve 4 out of the 6 problems. Each one carries the same number of points as any other.

## Problem IV.I

One method of tuning a Fabry-Perot cavity is to change the gas density in the space between the two mirrors (the index of most gases can be approximated by $n \approx 1 + \Delta$, where $\Delta$ is a small positive real number). Figure 14 is a sketch of transmission through the cavity of 0.6328-$\mu$m radiation when the gas pressure is changed. At a given pressure, the transmission peaks at 100% (point A). As the pressure increases, the transmission drops to a very small value, to re-appear at a higher pressure (point B). The spacing of the mirrors is fixed at $d = 1$ cm. Both mirrors are highly reflective and have the same (power) reflectance $R$.

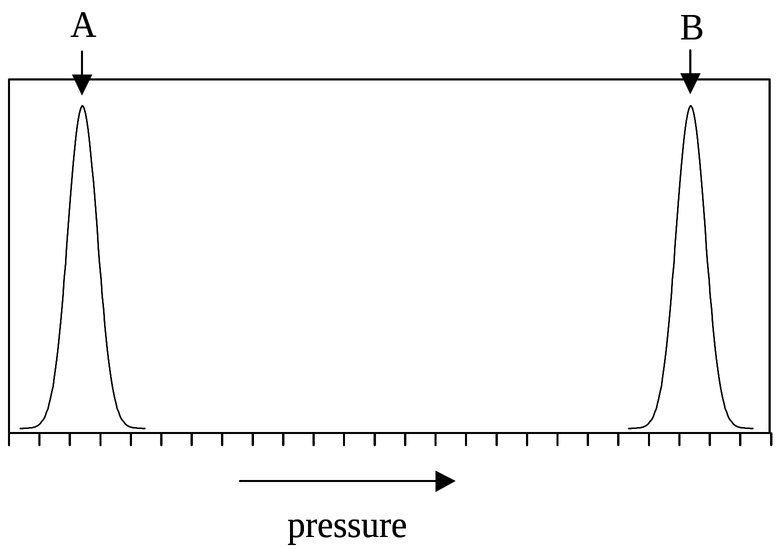



Figure 14: Fabry-Perot transmission showing two successive transmission peaks as the pressure is increased.

(a) (3 pts) By how much has the index of refraction changed between the points labelled $A$ and $B$?.

(b) (3 pts) By solving a rate equation for the photon number in the cavity, show that one can define a photon lifetime by

$$\tau \approx \frac{2\pi n d/c}{1 - R^2}.$$

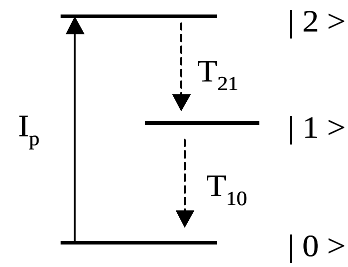
(c) (4 pts) Obtain (estimate) a numerical value for the photon lifetime of this cavity using the information given in the problem and the graph.

<u>Hint:</u> The finesse $F$ of a Fabry-Perot can be obtained by dividing the free spectral range by the full-width at half maximum of the transmission peaks:

$$F = \frac{\pi\sqrt{R}}{1 - R}$$

## Problem IV.2

Consider the energy level diagram of a laser material sketched in the figure. An external CW pump laser (frequency $\nu_p$, intensity $I_p$), tuned to the center of the 0-2 transition (cross section $\sigma_{02}$), pumps the atomic system (total number density $N$) with the goal of creating population inversion on the 2-1 transition (cross-section $\sigma_{21}$). The fluorescence lifetimes of levels 1,2 are $T_{10}$ and $T_{21}$, respectively.




(a) (4 pts) Formulate the rate equations including stimulated emission on the 2-1 transition.

(b) (3 pts) Neglect stimulated emission, assume steady-state, and solve for the maximum inversion possible (i.e., when the external laser is infinitely strong).

(c) (3 pts) Find the coefficient of the small signal gain for the 2-1 transition in terms of the pump intensity and the material parameters defined in the problem.

## Problem IV.3

A pump source creates an inversion of $\Delta N = 2 \times 10^{17}$ cm$^{-3}$. The laser parameters are:

Mirror reflectivity $R_1 = 0.90$, $R_2 = 0.80$,
Gain length $\ell_g = 10$ cm,
Cavity length $d = 20$ cm,
Index of refraction $n = 1$,
Lasing wavelength $\lambda = 700$ nm,
Gain cross section $\sigma = 10^{-18}$ cm$^2$,
Saturation intensity $I_s = 10$ kW/cm$^2$.

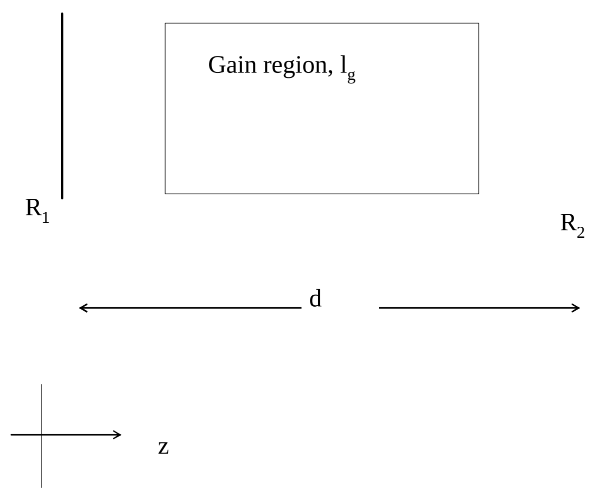



Figure 15: Laser.

(a) (2 pts.) What is the small signal gain $\gamma_0$?

(b) (2 pts.) Is this laser resonator above threshold?

At $t = 0$, an external source injects into the laser cavity an initial flux of photons at the lasing wavelength: $N_i = 10^8$ photons/cm$^2$. The photon field in the cavity is amplified to a photon flux $N_f$.

(c) (2 pts.) Sketch the laser output power as a function of time.

(d) (4 pts.) How much time does it take for the intracavity intensity $I$ to reach $I = 0.01 I_s$?

## Problem IV.4

Consider the laser system shown in the figure below. The following parameters are known: Stimulated emission cross section: $\sigma = 10^{-14}$ cm$^2$, Upper state lifetime: $\tau_2$ =20 ns, Broadening: $\Delta\nu$ =1/30 cm$^{-1}$, (collision broadened), Wavelength (line center): $\lambda_0$=600 nm, Beam Area : $A$=1 mm$^2$ (assume a uniform beam profile).

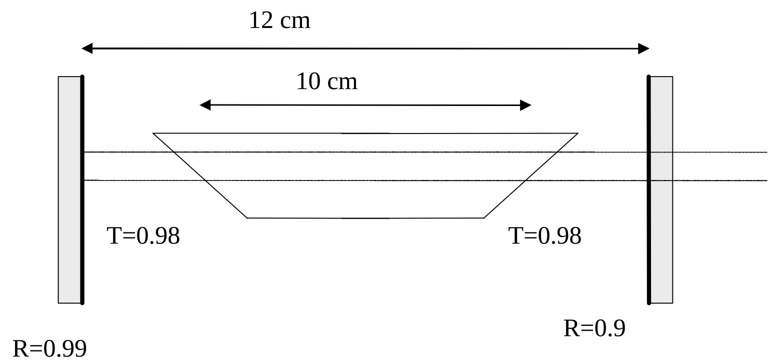



Figure 16: The reflection and transmission coefficients given in the figure are for intensities.

(a) (2 pts) Calculate the threshold gain ($\gamma_{th}$) for this laser. ( 2 pts.)

(b) (1 pt) What is the threshold population inversion density? (1 pts.)

(c) (3 pts) If the steady-state small-signal gain ($\gamma_0$) is $3\gamma_{th}$, calculate the output power (assume a high Q cavity). (3 pts.)

(d) (4 pts)Plot (qualitatively) the output power versus a small variation ($\Delta d$) in the cavity length for part (c). Assume TEM$_{q,0,0}$ modes only. Quantify your x-axis ( d).

## Problem IV.5

Consider the simple cavity sketched in Fig. 17, terminated by a flat mirror and a curved mirror of radius of curvature $R = 10$ cm. This laser can operate in a continuous mode within the stability limits of the cavity.

### (a) (3 pt)

In which range of the mirror spacing $d$ is this cavity stable?

### (b)

A nonlinear medium of thickness $\ell$ — i.e. a glass or crystal with an intensity dependent index of refraction — is inserted in the cavity. The laser beam, because of its radial intensity distribution, induces a lens in that medium with an intensity dependent focal length.

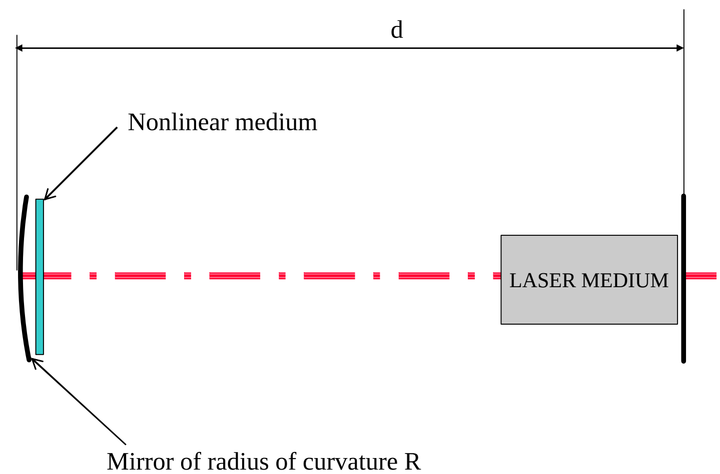



Figure 17: Cavity with a nonlinear medium against the curved mirror. The nonlinear medium is considered in part (b).

If the cavity length is slightly longer than the upper stability limit found in (a), the laser cannot operate. However, at higher intensity, the laser operation is possible if the nonlinear lensing induced by the intracavity laser light is sufficient to make the resonator stable. Assuming that the intensity of the intracavity radiation creates a lens of focal distance $|f| = 50$ cm:

1. **(1 pt)** what is the sign of the lensing (the sign of $f$) that will make the cavity stable?
2. **(3 pt)** Find the condition for the cavity length $d$ that will make this cavity stable, considering the nonlinear lensing $|f| = 50$ cm (make a thin lens approximation)
3. **(3 pt)** As the intensity is increased further, $|f|$ decreases. Is there a minimum value of $|f|$) which will cause this cavity to reach a stability limit? What is that value?

## Problem IV.6

### (a) (3 pts)

Derive the expression for the field transmission of a symmetric (identical mirrors) Fabry-Perot:

$$\mathcal{T}(\Omega) = \frac{(1-R)e^{-ikd}}{1-Re^{i\delta}}$$

where

$$\delta(\Omega) = 2\varphi_r - 2k(\Omega)d$$

is the total phase shift of a round-trip inside the Fabry-Perot, including the phase shift $\varphi_r$ upon reflection on each mirror.

We will be considering a Fabry-Perot of thickness $d = 1.52$ mm, filled with a dielectric of index $n = 3.6$. The phase shift upon reflection $\varphi_r$ is zero. One end mirror is on a piezoelectric translator, so that one cavity mode is centered at 800nm.

### (b) (1 pt.)

Calculate the free spectral range (in frequency and in wavelength) of that cavity.

### (c) (1 pt.)

Calculate the reflection coefficients $R_1 = R_2$ of the end mirrors of that cavity, to achieve a bandwidth of one mode (FWHM of a transmission peak) equal to $\Delta\lambda_{1/2} = 0.1$ nm.

### (c) (1 pt.)

All the parameters of the Fabry-Perot being thus determined, plot the intensity transmission versus wavelength $T(\lambda)$, including the three transmission modes around 800 nm (wavelength in vacuum). Indicate on the plot the values of the maximum and minimum transmission, and the corresponding wavelengths.

### (d) (2 pts)

The dielectric in the cavity serves as matrix for three types of *non-interacting* gain particles $A$, $B$ and $C$. Each has a Lorentzian gain shape with a bandwidth of 0.02 nm, centered respectively at

- 799.8 nm for $A$,
- 800 nm for $B$, and
- 800.2 nm for $C$.

Sketch the overlap of the cavity modes with these three gain media.

### (e) (2 pts)

Qualitatively sketch the output spectrum of this laser. What would the output be if, instead of the non-interacting particles $A$, $B$ and $C$, the medium inside the cavity were an homogeneously broadened gain medium, covering the bandwidth of the three modes? Explain.