Optical Sciences and Engineering Qualifying Examination 2007
Electromagnetics
Please attempt 4 of the following 6 problems. Please skim all problems before beginning, so as to use your time wisely: ALL PROBLEMS COUNT EQUALLY, BUT ALL MAY NOT BE EQUALLY DIFFICULT FOR YOU.

Begin each problem on a new page. Staple all pages of each problem together. Put your ID# on each page.

1. Electromagnetic Waves and Conductors

A monochromatic plane wave of radiation of angular frequency ω is incident from vacuum at the plane surface of a medium of real electrical conductivity σ and real permittivity ϵ.

(a). Show that the interaction of the monochromatic wave with the medium may be described accurately in terms of an effective (but complex) index of refraction. Express this index in terms of the given quantities and natural constants, and briefly describe the microscopic basis of the different parts of the expression. What is the physical significance of the complex nature of the index? [18 pts]

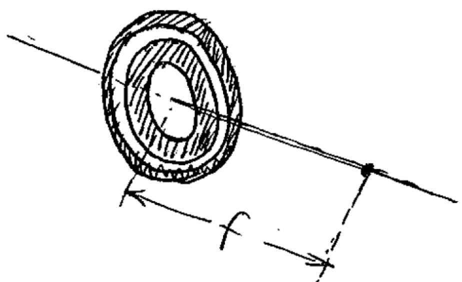
(b). What is the characteristic depth within the medium to which the incident wave will penetrate if it is incident at angle θ to the surface normal? Do not make any assumptions about the ratio, $\epsilon\omega/\sigma$, in order to simplify your expressions. [9 pts]

(c). Now assume the above ratio to be large compared to 1 and the thickness of the medium to be finite but large compared to the characteristic depth computed in part (b). Under these assumptions, calculate the coefficient of power transmission of the incident wave through the finite medium into the vacuum on the other side. [18 pts]

---

2. Fresnel Zones

The most elementary zone plate is comprised of alternating fully transparent and fully opaque annuli, as shown in the figure. Let us consider the design of a thin zone plate that serves as a lens of focal length f (the focus being a point on the central axis on the transmitted side where a plane wave normally incident on such a lens would yield maximum intensity).



(a). Based purely on considerations of light rays and optical phases, show that if the circular zones of the zone plate are chosen to have radii ρ_n such that $\sqrt{\rho_n^2 + f^2} - f = n\lambda/2$, then the corresponding rays from the transparent zones will interfere constructively at the focus. Here $\lambda = 2\pi/k$ is the wavelength of the incident plane wave. Take the central zone of the plate to be fully transparent. [13.5 pts]

(b). Consider now this problem from the perspective of scalar diffraction theory. Making only the assumption that the following Kirchoff integral over the transparent parts of the plate describes the diffracted field:

$$\psi(\vec{x}) = \frac{\psi_0}{i\lambda} \int d^2x' \frac{\exp(ik|\vec{x} - \vec{x}'|)}{|\vec{x} - \vec{x}'|},$$

determine the magnitude and phase of the diffracted field at the focus for a zone plate that has N transparent zones in all. Show that results of part (a) are consistent with this calculation. (*Hint:* The use of polar coordinates should greatly simplify the above surface integral, reducing it to a form in which the angular part of the integral is trivially done while its radial part requires a simple substitution, $u = \sqrt{\rho'^2 + f^2}$, before it too can be simply evaluated.) [18 pts]

(c). How do the results of part (b) change when all of the zones except the central one are made opaque? Also, show in this case that for sufficiently large f there are additional foci on the optical axis where intensity maxima are observed. How far from the plate are these foci located? [13.5 pts]

3. Radiation

A neutral silver atom in its ground state can be regarded as an elementary magnetic dipole corresponding to spin 1/2. When such an atom is placed in a static uniform magnetic field, its magnetic dipole moment vector precesses about the magnetic field direction, taken to be along the z axis, with the following time-dependent form:

$$\vec{M}(t) = m_0(\hat{x}\sin\theta_0\cos\omega t + \hat{y}\sin\theta_0\sin\omega t + \hat{z}\cos\theta_0),$$

where the moment m_0, the tip angle, θ_0, and the precession frequency, ω, are all real constants. Consider radiation from such a precessing magnetic dipole in the framework of classical electromagnetic theory.

(a). In the small-angle limit, $\theta_0 << 1$, write the real dipole-moment vector, $\vec{M}(t)$, as $\text{Re}[\vec{m}\exp(-i\omega t)]$ plus a constant, where $\text{Re}[A]$ denotes the real part of A. Express the time-independent (but complex) vector $\vec{m}$ in terms of the given quantities. [9 pts]

(b). Using the standard formulas for radiation by small sources, write down the angular distribution of the time-averaged far-field power radiated by the oscillating loop in the small-source limit, $a << 2\pi c/\omega$. [18 pts]

(c). What is the polarization of the radiation emitted along (i) the z axis and (ii) the x axis? Explain your results physically. [9 pts]

(d). Now imagine bringing a perfectly conducting plane close to the precessing magnetic dipole. One can use the method of images to solve this radiation problem. What is the (time-dependent) value of the image magnetic dipole moment vector? Justify your answer. [9 pts]

4. Conservation of Charge

a) From first principles, derive an expression for the conservation of charge (also known as the continuity equation) for a medium with uniform conductivity and permittivity.
b) Derive an expression for the continuity equation for a medium with a nonuniform conductivity.
c) Using Gauss's Law as applied to a medium with a nonuniform permittivity, and combining it with your result for b), derive a generalized expression known as the *charge relaxation equation* where $\tau_e = \varepsilon/\sigma$ is the relaxation time.

5. Polarization

Two different electromagnetic waves, $\underline{E}_1$ and $\underline{E}_2$, given by

$$\underline{E}_1(t) = \left[\hat{a}_x + \hat{a}_y\right] E_0 \cos(\omega t)$$
$$\underline{E}_2(t) = \left[\hat{a}_x - \hat{a}_y\right] E_0 \cos((\omega + \Delta\omega)t)$$

respectively, where $\Delta\omega << \omega$, are incident on a quarter wave plate.

a) Find an expression for the electric field of the total wave ($\underline{E}_1 + \underline{E}_2$) after passing through the quarter wave plate.

b) What is the polarization?

c) What is the sense of rotation (RH/LH) and the frequency of rotation?

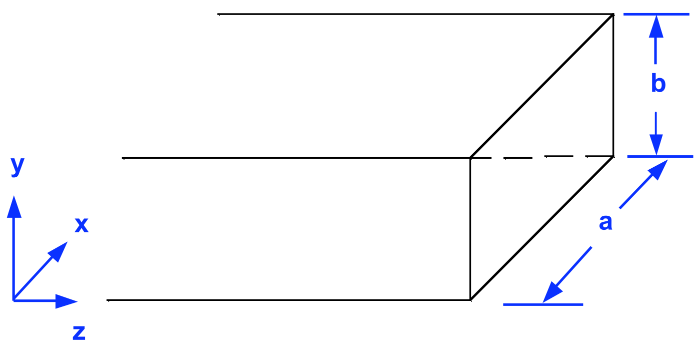
You may find the following helpful:

$$\cos a + \cos(a+b) = 2\cos\left(a + \frac{b}{2}\right)\cos\left(\frac{b}{2}\right) \qquad \sin a - \sin(a+b) = 2\cos\left(a + \frac{b}{2}\right)\sin\left(\frac{b}{2}\right)$$

6. Huygen's Principle

We wish to find the far field electric field radiated from the open end (aperture) of a rectangular waveguide of dimension a $\times$ b, as shown below. The mode propagating in the waveguide is TE_{10}, with E-field given by

$$E_y = \hat{y}E_0 \sin\left(\frac{\pi x}{a}\right) e^{j(\omega t - \beta z)}$$



a) Write down in words the steps you will use to solve this problem. Make any assumptions that are reasonable, and state them.

b) Write down approximate integral expressions for the appropriate far field vector potential (A or F) in terms of the waveguide electric field given above. YOU DO NOT HAVE TO DO THE INTEGRALS.

c) Write down approximate expressions for the far field electric field components E_r, E_θ, E_ϕ in terms vector potential in part b (A or F). YOU DO NOT HAVE TO DO THE INTEGRALS.

d) Describe the expected form of the solution for E in the far field, i.e. the polarization, and functional dependence in the x, y, and z directions. What optical phenomenon is at work here?

General Optics Qualifying Examination – 2007
Each problem counts equally. Attempt all problems.
Please begin each problem on a new page; indicate the problem number. Put your banner ID# on each page. Staple the pages for each problem together.

1. A red HeNe laser emitting 1 mW in a beam of 1 mm diameter is aimed at the Moon. For the Earth-Moon distance use 300 000 km.

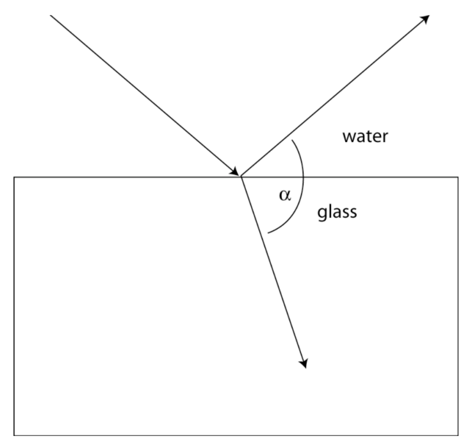
a) What is the **approximate** diameter of the illuminated region on the moon's surface? (Don't guess – estimate.)

b) **Estimate** the number of photons per second per m^2 that strike the surface of the Moon in the beam center. What is the intensity in the beam center? Neglect losses.

2. Draw a ray diagram for a refracting telescope, to observe distant planets. What is different, what is similar in optical binoculars?

3. A cylindrical cell (radius R = 5 mm, path length 1 cm) is filled with 10^{14} atoms of an ideal gas. A weak, quasi-monochromatic laser probes an absorption transition at resonance that has a cross section of 10^{-14} cm^2. Calculate the fraction (in percent) of incident laser power that is absorbed while passing through the cell. Neglect line broadening.

4. Natural (unpolarized) light in water (n=1.33) is incident on a plate of glass (n=1.5). The reflected light is completely polarized.



a) What is the angle α shown in the figure, between the reflected and transmitted beams?

b) What is the angle of incidence?

c) What is the polarization orientation for the polarized reflection?

5. The irradiance of sunlight above the earth's atmosphere is the "solar constant", 1360W/m^2. What is the force due to solar radiation on a perfectly reflecting "solar sail" of total area 4200m^2 oriented normal to the rays of sunlight and at the same distance from the sun as the earth?

6. A He-Ne laser operates at a frequency of 4.74×10^{14}Hz, with line width □□ = 1.5GHz. The light travels between plane mirrors separated by 30cm, forming an optical cavity. How many longitudinal modes are possible in this frequency range? Roughly how large are the mode numbers n of these modes?

7. If the index of refraction of an ideal gas is 1.005 at 1 atm. What is the index of refraction at 2 atm? At 100 atm? (Assume the gas remains ideal.)

8. The sodium doublet lines at 590 nm are separated by 0.4 nm. You have a 10-cm (square) grating available with 2000 lines per mm. Design (sketch) a spectrometer to resolve the doublet. Assume a sodium point emitter. You have pinholes, lenses and detectors at your disposal. Be as specific as possible in your design and explain whether or not you can actually resolve the doublet.

9. A monochromatic laser beam of wavelength □ is split. The resulting two beams are recombined under an angle □ on a screen. (a) Assuming plane waves, what do you expect to see? Quantify your answer using the parameters given? (b) Now consider that the frequency of one beam is shifted by 1 Hz before superposition with the second beam. What do you observe on the screen and why?

10. Give an example of an optical set-up demonstrating frustrated total internal reflection, and explain the physical cause of the phenomenon.

11. Describe qualitatively all three of the following:
a) Pockels effect
b) Kerr effect
c) Faraday effect
Use sketches and simple equations as appropriate. Comment on whether each effect is a non-linear optical effect; the order of the effect; the properties (symmetry, etc) of any material must have to exhibit the effect.

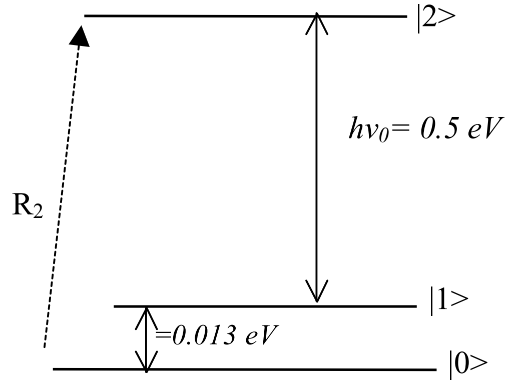
12. Describe the principles of operation of *either*
a Fourier transform infrared (FTIR) spectrometer
or
a Vertical cavity surface emitting semiconductor laser
or
a lock-in amplifier

Optical Sciences and Engineering Qualifying Examination Lasers 2007
Answer all questions. Begin each question on a new sheet of paper. Label each sheet with your ID# and problem number. Staple each problem separately. Good luck!

1. A homogeneously (pressure) broadened gaseous system is described by the energy diagram below. The following parameters are known:

- Dephasing time T_2 ($=1/g(\nu_0)$) = 10 ns
- Spontaneous lifetime τ_{sp}=10 msec
- Degeneracies: $g_2=3$, $g_1=1$, $g_0=1$, $n=1$
- $N_{total}=N_0+N_1+N_2=10^{18}$ cm^{-3}
- Temperature T=300 K

R_2 is the pump rate.



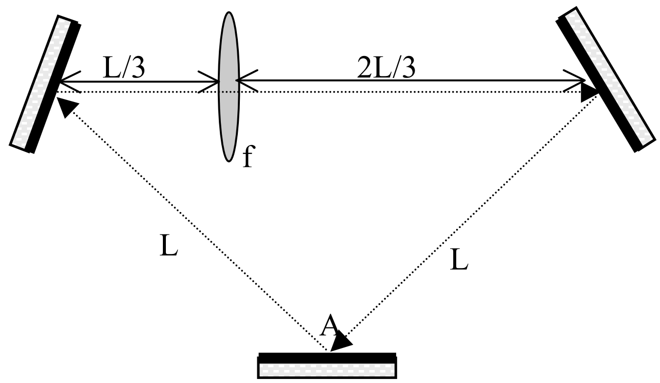
(a) What is the gain cross section at $\nu=\nu_0$? (2 points)

(b) At thermal equilibrium, the population of state |2> may be taken as zero, while that in state |1> may not be. Explain why with a quick estimation.

(c) With $R_2=0$, and assuming thermal equilibrium, what is the absorption coefficient $\alpha(\nu_0)$? Note: $\alpha(\nu_0)= -\gamma(\nu_0)$, where γ is the small signal gain coefficient.

(d) Under steady-state conditions, calculate the number density N_2 required to make $\alpha(\nu_0)= -\gamma(\nu_0)=0$. Assume Boltzmann equilibrium between levels |0> and |1>. Note: it is not necessary to solve the rate equations to do this.

2. Consider the ring cavity with flat mirrors shown below:



(a) Starting from point A and assuming clockwise propagation, write the matrix product needed to obtain the ABCD matrix of the cavity.

(b) Where is the beam waist in the cavity? Repeat (a), but start from the position of the beam waist.

(c) Find the ABCD system matrix of the cavity using (a).

(d) Find the values of f for which the cavity is stable.

3.(a) Explain each of the following line broadening mechanisms in a sentence or two. Which mechanisms are inhomogeneous and which are homogeneous?

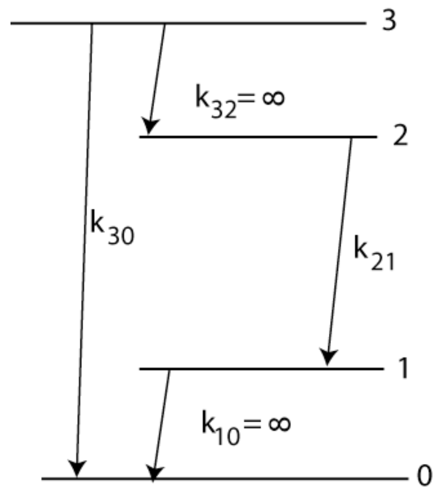
• Lifetime broadening
• Doppler broadening
• Pressure broadening

(b) Consider a laser operating at steady-state. Make a sketch of the gain as a function of frequency in a laser medium, for a homogeneously broadened laser medium (laser 1) and for an inhomogeneously broadened medium (laser 2). Assume there are six longitudinal modes that are above threshold at laser turn-on.

Note that, experimentally, you could measure the gain by sending a weak, tunable, monochromatic probe beam through the laser medium.

(c) Suppose a laser is mode-locked and is lasing on six longitudinal modes. What is the **approximate** ratio of the pulse duration to the round-trip time for this mode-locked laser?

4. Consider the four level system shown. The rates of spontaneous transitions are shown: any transitions not shown have a rate of zero. A pump laser is tuned to the 0-3 transition. The cross section for pumping is $\square_p$ and for stimulated emission between levels 2 and 1 is / /.



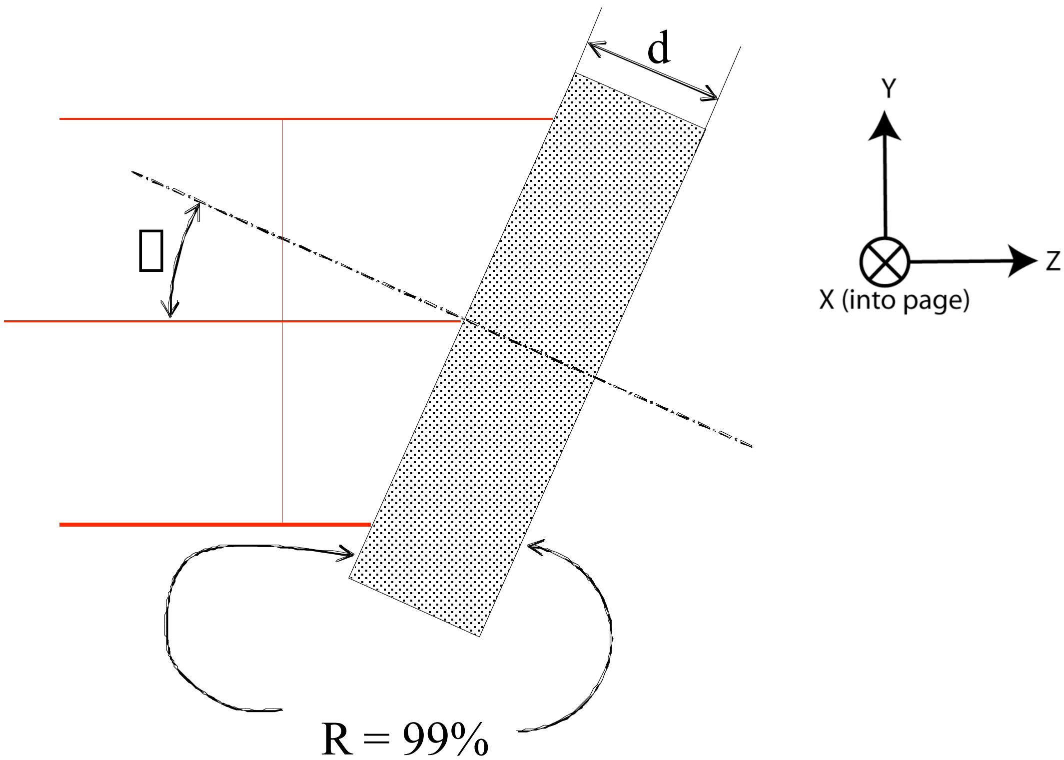
(a) Write the rate equation for dN_2/dt, in terms of the number densities N_0, N_1, N_2, N_3 and the pump I_p and laser I intensities. What are the values of N_3 and N_1 for this model?

(b) By solving for the steady state population inversion between states 2 and 1, find the small signal gain and I_s, the saturation intensity for the 2-1 transition. (Assume homogeneous broadening.)

5. A parallel glass slab of thickness *d*, as shown below, is coated on both faces with a 99% (intensity) reflectivity coating. A *plane wave* at 633 nm is incident from the left, at an angle of incidence θ. The thickness *d* is 10,000.00 μm, and the index of refraction of the glass is 1.5. Neglect phase shifts on reflection.

(a) Complete the sketch (show transmitted and reflected beams, and the wavefronts)

(b) For small θ, find the angles at which the transmission is maximum. Make (and justify) appropriate approximations. Is the angular spacing between maxima constant?



Let us choose an angle of incidence of $\theta = 30^{\circ}$. Instead of a plane wave, assume an incident beam with wavelength $\lambda = 633\ nm$ with a Gaussian intensity distribution in *y* with a waist $w_0 = 1\ mm$. You may ignore any variation in the intensity with *x* (i.e. assume the intensity does not vary with *x*.)

(c) Is there a qualitative difference between this problem and that in the previous part? If you have no clue, proceed to the next parts of this question and enlightening might come.

(d) Write down the Gaussian field distribution in the *y* direction.

(e) Calculate the spatial Fourier transform $E(k_y)$ of that distribution.

(f) Re-write the transmission function of the Fabry-Perot with the component k_y as variable, remembering that k_y is orthogonal to k.

(g) Write an expression for the transmission (in k_y space) of the Fabry-Perot.

(h) Describe what you see on a screen placed at the right of the plate.

Advanced Optics 2007

Diffraction and Interference

1. First, consider diffraction from a single slit of width w = 50 μm in air, illuminated by a plane wave (λ = 500 nm).

a. Derive the far-field intensity distribution, as a function of angle from the center of the slit, θ, normalized to $\theta = 0$.

b. At what angle is the first zero of intensity found?
(If you were unable to do [1a], you may still be able to solve [1b] using path length arguments.)

c. Consider now both the near and the far field. An ideal, point-like photodetector is moved toward the slit, on the centerline, starting from far away. Is there any point z at which the intensity *decreases* as the detector is moved *closer* to the slit? Explain why or why not.

If so, at what distance z does this first occur?

2. Now, consider a two-slit (Young's) experiment.

a. Assume the slits have a separation d=500 μm and width w<<λ. Derive the far-field intensity distribution as a function of angle from the center of the slit, θ. (You may neglect the slit width.)

b. Your answer to [2a] should show that there are interference fringes from the two slits. In a real two-slit experiment, it is observed that the fifth fringe away from the center of the pattern is "missing". Explain in words why this fringe is missing.

c. What was the width of the slits used in the real experiment?

3. Michelson Interferometer & Coherence
A laser produces an output described by the electric field

$$E(t) = A_0(t)\cos(2\pi\nu_0 t + \phi(t))$$

where $A_0(t)$ is the time-dependent field envelope, ν_0 is the center frequency of the laser, and $\phi(t)$ is a time-dependent phase.

One studies the light by sending it through a balanced Michelson interferometer, which splits the beam, introduces a delay τ into one part by translating one of the arms, and then recombines the two beams. The signal at the detector (called an *interferogram*) is

$$I(\tau) = \tfrac{1}{2}c\varepsilon_0\left\langle \left|E(t) + E(t+\tau)\right|^2 \right\rangle_t$$

(The detector automatically performs the average over time, because its time response is slow compared with any variation in $A_o(t)$.)

Multiplying out the square, one finds three terms. The cross term, or interference term, is proportional to

$$g(\tau) = \langle E(t)E(t+\tau)\rangle_t = \frac{1}{T}\int_{-T/2}^{T/2} E(t)E(t+\tau)dt$$

where T is the detector averaging time.

Sketch **(approximately)** the interferogram $I(\tau)$ for $-4<\tau\nu_0<4$ assuming the following conditions:

a. A_0 and ϕ are constant (i.e. continuous wave 'cw' monochromatic light)

b. A_0 represents a train of Gaussian pulses, each with a full width at half maximum of $\tau_p \sim 3/\nu_0$. ϕ is constant with time.

c. A_0 is constant (cw) but the phase ϕ jumps randomly (i.e. to a random value between $-\pi$ and $+\pi$) after a coherence time $\tau_c \sim 3/\nu_0$.
Can you distinguish the interferogram in (c) from that of a pulsed source with unknown $A_0(t)$?
Can you use $g(\tau)$ to measure the pulse width of an arbitrary laser source? Explain.

d. Derive the relationship between $g(\tau)$ and the Fourier transform $\tilde{E}(\omega)$ of the field E(t), for a pulse.
Hints:
You may take the limits of the integral to be $\pm\infty$, since the pulse is short.
Also, $E(t) = E^*(t)$, since the field is real.

4. Polarization

An experiment is to be designed using a short pulse laser. The laser output wavelength is centered at 900 nm; the spectrum extends from 750 nm to 1050 nm. The laser output is linearly polarized, E-field vertical.

a. It is desired to convert the linearly polarized pulses into circularly polarized light. For this purpose, a quarter wave plate is to be used. Two choices are available– a "zero order" QWP and an 80th order QWP. (Both are single crystals; assume they are dispersionless) Which should be used, or are they equivalent in this application? Why?

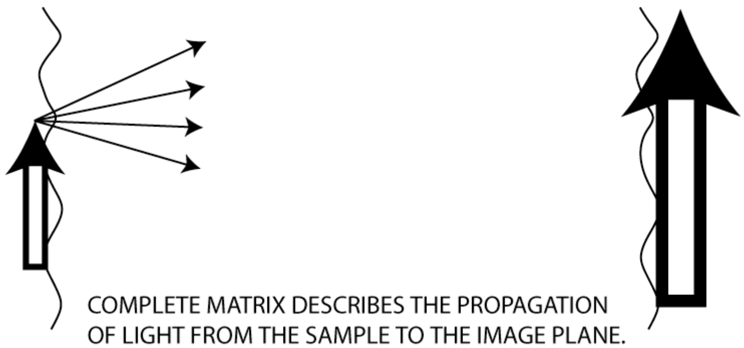
b. The optimum choice of QWP is placed in position. How should it be oriented with respect to the linear input polarization, to best produce circularly polarized light at 900 nm?

c. A linear polarizer is then used to analyze the output of the QWP oriented to give circularly polarized light at 900 nm. A narrow band color filter is also used, after the analyzer. Plot the output of a detector placed behind the analyzer/filter for an 800 nm narrow band filter, as a function of **angle** of the analyzer. Be as quantitative as possible.

5. Ray / Matrix Optics

a. Using a sketch and a few sentences and equations, show that the matrix that propagates (or translates) a paraxial ray $\begin{bmatrix} y \\ \theta \end{bmatrix}$ a distance L is $\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}$.

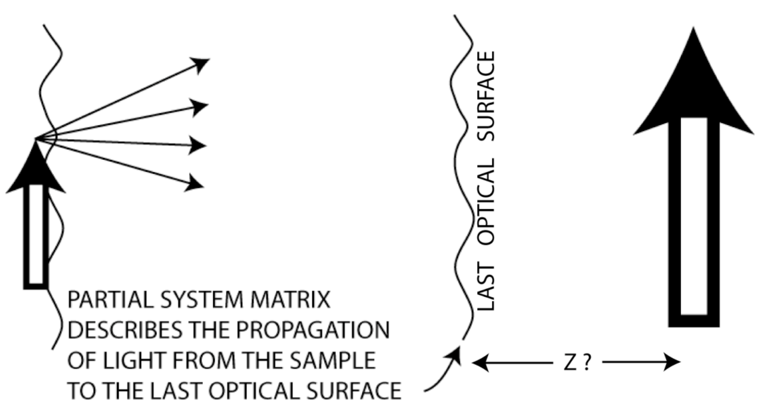
b. Consider a paraxial optical microscope system, configured to produce a sharp image on a CCD camera chip. What does the system matrix do to *all* of the paraxial rays emanating from the same point on the sample?



c. The microscope under discussion has a *partial* system matrix:

$$\begin{bmatrix} 8 & -0.2 \\ 1 & +0.1 \end{bmatrix}$$

describing propagation of light from the sample plane to the last optical surface, as shown. Units (where required) are meters. How far behind the last optical surface should a CCD chip be placed to take a sharp image?



d. What is the magnification of the microscope?

Formulae and Constants

$e = 1.6\times10^{-19}\,\mathrm{C}$
$h = 6.6\times10^{-34}\,\mathrm{J\cdot s}$
$c = 3\times10^{8}\ \mathrm{m/s}$
$hc = 1240\ \mathrm{eV\cdot nm}$
$\varepsilon_0 = 8.85\times10^{-12}\ \mathrm{C^2/Nm^2}$
$\mu_0 = 4\pi\times10^{-7}\ \mathrm{N/A^2}$
$k_B T = 26\ \mathrm{meV}$ at $T = 300K$

$$k = n\frac{\omega}{c} = \frac{2\pi n}{\lambda_0}$$

Irradiance: $I = \langle S\rangle = \dfrac{nc\varepsilon_0}{2}E_0^2$

Snell's Law: $n_i \sin\theta_i = n_t \sin\theta_t$

Lens Transformation of a Gaussian beam: $\dfrac{1}{R_{out}} = \dfrac{1}{R_{in}} - \dfrac{1}{f}$

Lens-maker's formula: $\dfrac{1}{f} = (n-1)\left(\dfrac{1}{R_1} - \dfrac{1}{R_2}\right)$

Fourier Transforms

$$f(t) = \int_{-\infty}^{\infty} g(\omega)e^{-i\omega t}d\omega \qquad g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt \qquad \text{Gaussian}: f(t) = e^{-at^2} \leftrightarrow g(\omega) = \sqrt{\frac{\pi}{a}}e^{-\omega^2/4a}$$

$$f(t) = \begin{cases} e^{-i\omega_0 t}, & -\frac{\tau_0}{2} < t < \frac{\tau_0}{2} \\ 0, & \text{elsewhere}\end{cases} \qquad g(\omega) = \frac{\sin[(\tau_0/2)(\omega-\omega_0)]}{\pi(\omega-\omega_0)} = \frac{\tau_0}{2\pi}\left\{\frac{\sin[(\tau_0/2)(\omega-\omega_0)]}{[(\tau_0/2)(\omega-\omega_0)]}\right\}$$

Gaussian Beams:

$$\frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda}{\pi w^2(z)}, \qquad w^2(z) = w_0^2\left[1+\frac{z^2}{z_0^2}\right], \qquad R(z) = z\left[1+\frac{z_0^2}{z^2}\right], \qquad z_0 = \frac{\pi n w_0^2}{\lambda_0}$$

Fresnel:

$$r_\parallel = \frac{n_t\cos\theta_i - n_i\cos\theta_t}{n_t\cos\theta_i + n_i\cos\theta_t} = \frac{\tan(\theta_i-\theta_t)}{\tan(\theta_i+\theta_t)} \qquad r_\perp = \frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t} = -\frac{\sin(\theta_i-\theta_t)}{\sin(\theta_i+\theta_t)}$$

$$t_\parallel = \frac{2n_i\cos\theta_i}{n_i\cos\theta_t + n_t\cos\theta_i} = \frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_i+\theta_t)\cos(\theta_i-\theta_t)} \qquad t_\perp = \frac{2n_i\cos\theta_i}{n_i\cos\theta_i + n_t\cos\theta_t} = \frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_i+\theta_t)}$$

when there is total internal reflection at an air or vacuum interface:

$$r_\parallel = \frac{\cos\theta_i - in_i\sqrt{n_i^2\sin^2\theta_i - 1}}{\cos\theta_i + in_i\sqrt{n_i^2\sin^2\theta_i - 1}} \qquad r_\perp = \frac{n_i\cos\theta_i - i\sqrt{n_i^2\sin^2\theta_i - 1}}{n_i\cos\theta_i + i\sqrt{n_i^2\sin^2\theta_i - 1}}$$

Fabry-Perot: *Field* Transmission (length=d): $\tau = \dfrac{(1-R)e^{-i\vec{k}\cdot\vec{d}}}{1-Re^{i\delta}}$

where the total round-trip phase shift, including the phase shift δ_r at each mirror, is

$\delta = 2\delta_r - 2\vec{k}\cdot\vec{d} = 2\delta_r - 2kd\cos\theta$

Blackbody Radiation (energy density): $\rho(\nu)d\nu = \dfrac{8\pi n^3 h\nu^3 d\nu}{c^3}\dfrac{1}{e^{h\nu/kT}-1}$

Thermal equilibrium $\dfrac{N_b}{N_a} = \dfrac{g_b}{g_a}\exp\left(-\dfrac{E_b - E_a}{k_B T}\right)$

Waveguides

$$\vec{H}_t = \frac{\pm 1}{Z}\hat{z}\times\vec{E}_t \qquad Z = \begin{cases} \dfrac{k}{\epsilon\omega} & \text{(TM)} \\[2mm] \dfrac{\mu\omega}{k} & \text{(TE)} \end{cases} \qquad \begin{Bmatrix} \vec{E}_t \\ \vec{H}_t \end{Bmatrix} = \pm\frac{ik}{\gamma^2}\nabla_t\psi \qquad (\nabla_t^2+\gamma^2)\psi = 0 \qquad \gamma^2 = \mu\epsilon\omega^2 - k^2 \qquad \psi e^{\pm ikz} = \begin{Bmatrix} E_z\ , & \psi|_S \\ H_z\ , & \dfrac{\partial\psi}{\partial n}\Big|_S \end{Bmatrix} = 0$$

Gain in a two-level system: $\gamma(\nu) = \sigma(\nu)\left(N_2 - \dfrac{g_2}{g_1}N_1\right)$ Gain cross section: $\sigma(\nu) = A_{21}\dfrac{\lambda^2}{8\pi n^2}g(\nu)$

Lineshape Normalization: $\int g(\nu)d\nu = 1$ Beer's Law: $\dfrac{1}{I}\dfrac{dI}{dz} = -\alpha(I) + \gamma(I)$

Gain or absorption saturation in a homogenously-broadened system:

$$\gamma(I) = \frac{\gamma_0}{1+I/I_s} \quad \text{or} \quad \alpha(I) = \frac{\alpha_0}{1+I/I_s} \qquad I_s(\nu) = \frac{h\nu}{\sigma(\nu)\tau_2}$$

Einstein's relation: $\dfrac{A_{21}}{B_{21}} = \dfrac{8\pi n^3 h\nu^3}{c^3}$ $\quad g_2B_{21} = g_1B_{12} \quad$ $\dfrac{N_2}{N_1} = \dfrac{g_2}{g_1}e^{-(E_2-E_1)/kT}$

Maxwell:

$$\nabla\cdot\vec{D} = \rho$$
$$\nabla\times\vec{H} - \frac{\partial\vec{D}}{\partial t} = \vec{J}$$
$$\nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$$
$$\nabla\cdot\vec{B} = 0$$
$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = 0$$

$$\vec{F} = q(\vec{E}+\vec{v}\times\vec{B})$$
$$\vec{E} = -\nabla\Phi$$
$$\nabla^2\Phi = -\rho/\epsilon_0$$

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{Ne^2}{\epsilon_0 m}\sum_j f_j(\omega_j^2 - \omega^2 - i\omega\gamma_j)^{-1} \quad \text{with} \quad \sum_j f_j = Z$$

High frequency limit: $\dfrac{\epsilon(\omega)}{\epsilon_0} \approx 1 - \dfrac{\omega_p^2}{\omega^2}$ with $\omega_p^2 = \dfrac{NZe^2}{\epsilon_0 m}$

Ionosphere propagation $\parallel$ to $\vec{B}_0$: $\dfrac{\epsilon_\mp}{\epsilon_0} = 1 - \dfrac{\omega_p^2}{\omega(\omega\mp\omega_B)}, \ \omega_B = \dfrac{eB_0}{m}$

Skin Depth: $\delta = \sqrt{\dfrac{2}{\mu_c\omega\sigma}}$

$$d\vec{B} = \frac{\mu_0}{4\pi}I\frac{(d\vec{l}\times\vec{x})}{|\vec{x}|^3}$$
$$\vec{S} = \tfrac{1}{2}(\vec{E}\times\vec{H}^*)$$

Dipole Radiation

$$\vec{H} = \frac{ck^2}{4\pi}(\hat{n}\times\vec{p})\frac{e^{ikr}}{r}$$
$$\vec{E} = \sqrt{\frac{\mu_0}{\epsilon_0}}\vec{H}\times\hat{n}$$
$$P = \frac{c^2k^4}{12\pi}\sqrt{\frac{\mu_0}{\epsilon_0}}|\vec{p}|^2$$