EQUATION SHEET

ELECTROMAGNETISM

Possibly Useful Formulas

- Relation of spherical coordinates, (r, θ, ϕ), to Cartesian coordinates:

$$x = r\,\sin\theta\,\cos\phi,\quad y = r\,\sin\theta\,\sin\phi,\quad z = r\,\cos\theta.$$

  Unit vectors:

$$\hat{r} = \sin\theta\,\cos\phi\,\hat{x} + \sin\theta\,\sin\phi\,\hat{y} + \cos\theta\,\hat{z};$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y};\quad \hat{\theta} = \hat{\phi}\times\hat{r}.$$

- Electric potential at position $\vec{r}$ due to a point electric dipole of moment $p\hat{z}$ located at the origin:

$$V(\vec{r}) = \frac{\vec{p}\cdot\vec{r}}{4\pi\epsilon_0 r^3}.$$

- Polarization induced in a dielectric sphere of dielectric permittivity, ϵ, by a uniform external field, $\vec{E}$:

$$\vec{P} = 3\epsilon_0\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)\vec{E},\quad \epsilon_r \equiv \frac{\epsilon}{\epsilon_0}.$$

- Time-averaged power radiated by an oscillating electric dipole:

$$P = \frac{\mu_0 |p|^2 \omega^4}{12\pi c}.$$

- Time-averaged power radiated by an oscillating magnetic dipole:

$$P = \frac{\mu_0 |m|^2 \omega^4}{12\pi c^3}.$$

- Electric and magnetic fields at point $\vec{r}$ due to an oscillating electric dipole of moment $\vec{p}\exp(-i\omega t)$ at the origin -

$$\vec{E} = \frac{e^{i(kr-\omega t)}}{4\pi\epsilon_0}\left\{k^2\frac{(\hat{r}\times\vec{p})\times\hat{r}}{r} + \left(\frac{1}{r^3} - \frac{ik}{r^2}\right)[3(\hat{r}\cdot\vec{p})\hat{r} - \vec{p}]\right\};$$

$$\vec{B} = \frac{k^2 e^{(kr-\omega t)}}{4\pi c\epsilon_0}(\hat{r}\times\vec{p})\left(\frac{1}{r} + \frac{i}{kr^2}\right);\ k = \frac{\omega}{c}; \hat{r} = \frac{\vec{r}}{r}$$

- Fresnel formulas for the amplitude reflection coefficient of a plane wave incident at a planar interface between two dielectrics:

$$r_\perp = \frac{n\cos\theta - n'\cos\theta'}{n\cos\theta + n'\cos\theta'};\quad r_\parallel = \frac{n'\cos\theta - n\cos\theta'}{n'\cos\theta + n\cos\theta'},$$

  where $\perp, \parallel$ refer, respectively, to polarizations perpendicular and parallel to the plane of incidence. The angles of incidence and refraction are θ and θ', and n, n' are the refractive indices of the medium of incidence and the medium of transmission, respectively.

EQUATION SHEET

OPTICS

>> **Ray tracing matrices:**

Thin lens: $\begin{bmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{bmatrix}$ Free space: $\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$ Curved interface: $\begin{bmatrix} 1 & 0 \\ \frac{n_1-n_2}{Rn_2} & \frac{n_1}{n_2} \end{bmatrix}$ (R: radius)

>> **Snell's law:** $n_i Sin\theta_i = n_t Sin\theta_t$ **Lens-makers's formula:** $\frac{1}{f} = (\frac{n_2-n_1}{n_1})(\frac{1}{R_1} - \frac{1}{R_2})$ (R_i: radius)

> **Grating equation:** $Sin\,\theta' - Sin\,\theta = m\frac{\lambda}{d}$

>> **Gaussian beam:** $\frac{1}{q} = \frac{1}{R(z)} + \frac{i\lambda_0}{n\pi w^2(z)}$ $w(z) = w_0^2\left(1 + \frac{z^2}{z_0^2}\right)$, $R(z) = z\left(1 + \frac{z_0^2}{z^2}\right)$, $z_0 = \frac{\pi n w_0^2}{\lambda_0}$

$$\frac{E(x,y,z)}{E_{m,p}} = H_m\left[\frac{\sqrt{2}\,x}{w(z)}\right] H_p\left[\frac{\sqrt{2}\,y}{w(z)}\right] \times \frac{w_0}{w(z)} \exp\left[-\frac{x^2+y^2}{w^2(z)}\right] \times \exp\left\{-j\left[kz - (1+m+p)\tan^{-1}\left(\frac{z}{z_0}\right)\right]\right\} \times \exp\left[-j\frac{kr^2}{2R(z)}\right]$$

>> **Gaussian beam propagation from point 1 to point 2 :** $q_2 = \frac{Aq_1+B}{Cq_1+D}$

>> **Fabry-Perot** *Transmisison* (when r_1=r_2=r): $\frac{E_t}{E_i} = \frac{(1-r^2)e^{-i\delta/2}}{1-r^2e^{-i\delta}}$, $\delta = 2kd$, $T = \left|\frac{E_t}{E_i}\right|^2 = \frac{(1-r^2)^2}{1+r^4-2r^2\cos(\delta)}$

r is *reflection coefficient* and ***R*** (=r^2) is *reflectance*. For asymmetric case ($r_1 \neq r_2$) : $r^2 \rightarrow r_1 \times r_2$

Finess: $F = \frac{\Delta\nu_{FSR}}{2\Delta\nu_{1/2}} = \frac{\pi r}{1-r^2}$, Photon life time: $\tau = \frac{\tau_r}{\delta_c} = \frac{Q}{\omega_0} = \frac{\tau_{RT}}{1-S} = \frac{2nd/c}{1-R^2}$, (*note that* $2\Delta\nu_{1/2}$ *is the FWHM*)

Free spectral range: $\Delta\nu_{FSR} = \frac{c}{2nd}$, Quality fator: $Q = \frac{\nu}{\nu_{FSR}} \times F = \frac{\nu}{2\Delta\nu_{1/2}}$

>> **Fringe visibility** (two path inteference with single source): $V = \frac{2|\gamma(\tau)|\sqrt{I_1 I_2}}{I_1+I_2}$, **γ(τ):** complex degree of temporal coherence

>> **Faraday cell rotation:** $\beta = VBd$, **Kerr effect:** $\Delta n = KE^2\lambda$, **Pokels effect:** $\Delta n = n^3\left(\frac{r_{eo}}{2}\right)E$

>> **Irradiance:** $I = \langle S \rangle = \frac{1}{2\times\sqrt{\mu_0/\epsilon}} E_0^2 = \frac{nc\varepsilon_0}{2} E_0^2$

>> **Fresnel equations:**

$$r_\parallel = \frac{\tan(\theta_i-\theta_t)}{\tan(\theta_i+\theta_t)} = \frac{n_t\cos\theta_i - n_i\cos\theta_t}{n_i\cos\theta_t + n_t\cos\theta_i} \qquad r_\perp = -\frac{\sin(\theta_i-\theta_t)}{\sin(\theta_i+\theta_t)} = \frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t}$$

$$t_\parallel = \frac{2\sin\theta_i\cos\theta_i}{\sin(\theta_i+\theta_t)\cos(\theta_i-\theta_t)} = \frac{2n_i\cos\theta_i}{n_i\cos\theta_t + n_t\cos\theta_i} \qquad t_\perp = \frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_i+\theta_t)} = \frac{2n_i\cos\theta_i}{n_i\cos\theta_i + n_t\cos\theta_t}$$

>> **Fresnel integral:** z=distance between aperture plane (ξ,η) and observation plane (x,y)

$$U_p(x,y) = \frac{ie^{-ikz}}{\lambda z} e^{(-ik/2z)(x^2+y^2)} \iint_{-\infty}^{+\infty} U_A(\xi,\eta) \times e^{(-ik/2z)(\xi^2+\eta^2)} \times e^{-i(2\pi/\lambda z)(x\xi+y\eta)} d\xi d\eta$$

>> **Phase transformation of a spherical lens:** $t_l(x,y) = exp\left[\frac{ik}{2f}(x^2+y^2)\right]$

>> **Fourier Transforms:** $\mathcal{F}\{g(x)\} = \int_{-\infty}^{\infty} g(x)e^{i2\pi(xf_x)}dx$, $\mathcal{F}^{-1}\{G(f_x)\} = \int_{-\infty}^{\infty} G(f_x)e^{-i2\pi(xf_x)}df_x$, $f_x = \frac{x}{\lambda z}$

(between space and spatial frequeny domain)

>> **Free space propagation** (*frequency domain*): $\mathcal{F}\{U(x)\} = \mathcal{F}\{U(\xi)\} \times exp(i\frac{2\pi^2}{k}zf_x)$

OPTICS (continued)

Jones matrices:

Linear polarizer : *TA horzontal:* $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$, *TA vertical* $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$

Phase retarders : $\begin{bmatrix}e^{i\delta_x} & 0\\ 0 & e^{i\delta_y}\end{bmatrix}$, *Rotator* : $\begin{bmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{bmatrix}$

LASER

Einstein coefficients: $\frac{A_{21}}{B_{21}} = \frac{8\pi n^3 h\nu^3}{c^3}$, $g_2 B_{21} = g_1 B_{12}$, $\frac{1}{\tau_{21,rad}} = A_{21}$

Natural linewidth : $g(\nu) = \frac{\Delta\nu}{2\pi\left[(\nu_0-\nu)^2+\left(\frac{\Delta\nu}{2}\right)^2\right]}$, $\Delta\nu = \frac{1}{2\pi}\left(\frac{1}{\tau_2}+\frac{1}{\tau_1}\right) = \frac{1}{2\pi}(A_1 + A_2)$, $A_2 = \sum_{j<2} A_{2j}$

Stimulated emission cross section: $\sigma(\nu) = A_{21}\frac{\lambda^2}{8\pi n^2} g(\nu)$

Gain (loss) for a two-level system: $\gamma(\nu) = \sigma(\nu)\left[N_2 - \frac{g_2}{g_1} N_1\right]$

Gain/Absorption: $\frac{dI_\nu}{dz} = \frac{\gamma_0 I_\nu}{1+\bar{g}(\nu)\left(I_\nu/I_s\right)}$

Saturation intensity: $I_s = \frac{h\nu}{\sigma(\nu)\tau_2}$

CONSTANTS

$h = 6.62\times10^{-34}$ J.s

$c = 3\times10^{8}$ m/s

$e = 1.6\times10^{-19}$ C

$k = 1.38\times10^{-23}$ J.K^{-1}

Instructions:

Answer all questions

Begin each question on a new sheet of paper!

Staple all pages together

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1-Laser light passes through two identical and parallel slits, 0.2 mm apart. Interference fringes are seen on a screen, perpendicular to the slits, 1 m away. The interference maxima are separated by 3.29 mm. what is the wavelength of the light used? (10 points)

2-The sodium doublet lines at 590 nm are separated by 0.4 nm. You have a 10 cm (square shape) grating available with 2000 lines per mm. Design (sketch) a spectrometer to resolve the doublet. Assume a sodium point emitter. You have pinholes, lenses and detectors at your disposal. Be as specific as possible in your design and explain whether or not you can actually resolve the doublet. (10 points)

3-Describe mode-locking and give typical characteristics of a mode-locked laser output. Give one example each for both active and passive mode-locking techniques. *Do not exceed 1 page*. (10 points)

4-Using the diffraction limited resolution of the human eye, find the longest distance at which one can distinguish the two headlights of an incoming car at night. Use reasonable assumptions for the car and human dimensions. (10 points)

5-Suppose the red line from a low pressure cadmium lamp is 648.3 nm gives visible fringes in a (single pass) Michelson interferometer when the movable mirror is moved through a distance between 0 (balanced interferometer) and 10 cm:

a) Explain in words why the fringes disappear if the mirror is moved farther.

b) What is the spectral width of the line (in nm)?

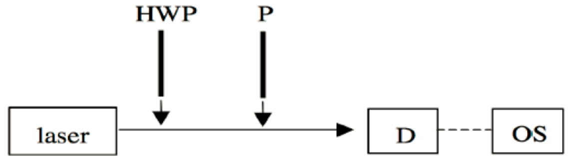
c) What is the coherence time of this line?

(15 points)

6-Consider the experimental setup sketched in the figure below. The laser source emits polarized beam of power P_0.

a) A linear polarizer (**P**) is inserted between laser and detector and rotates with an angular velocity ω. Sketch the trace that one sees on an oscilloscope (**OS**). Make sure you calibrate both axes using the quanitities given. Explain the behavior.

b) The polarizer is at a fixed orientation in the beam path with its transmission axis parallel to the polarization emitted by the laser. A half-wave plate (**HWP**) that rotates with an angular frequency ω is inserted between laser and polarizer. Sketch the oscilloscope trace. Make sure you calibrate both axes in terms of P_0 and . Explain the behavior.



(20 points)

7-A screen contains a small pattern, illuminated by 800 nm, light, which you want to project on a CCD image sensor with a magnification factor of 5. You are given a lens with a focal length of +20 cm and diameter of 5 cm, made of fused silica.

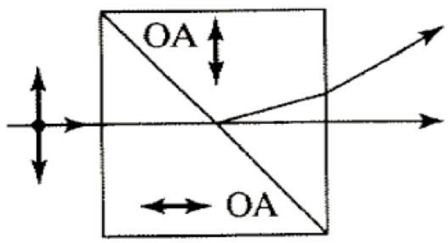
a) Find the relative positions of the lens and the CCD image sensor that will give you the sharpest magnified image.

b) The image sensor consists of closely packed 10 μm × 10 μm pixels. Is the resolution of the system limited by the CCD or by the lens? Why?

c) If we change the illumination to 400 nm, will the image still be sharp on the CCD? If not, in which direction should we move the CCD to get a sharp image? (assume normal dispersion for the lens)

(20 points)

8-Unpolarized light enters a Sernamont prism as shown below. On the figure indicate the polarization of the two emerging beams. Is the material used to make the prism a poitive or negative uniaxial crystal? (10 points)



9-a) What is the physical meaning of phase velocity and group velocity? (10 points)

b) Describe two methods for Q-switching a laser. (10 points)

c) Explain the operation of a semiconductor photodiode. (15 points)

d) Why does your grandmother squint when treading a needle? (5 points)

e) Explain why Brewster windows are sometimes used in lasers. (5 points)

f) What should be the orientation (with respect to the line connecting the eyes) of the transmission axis of the polarized sunglasses?Why? (5 points)

(total: 50 points)

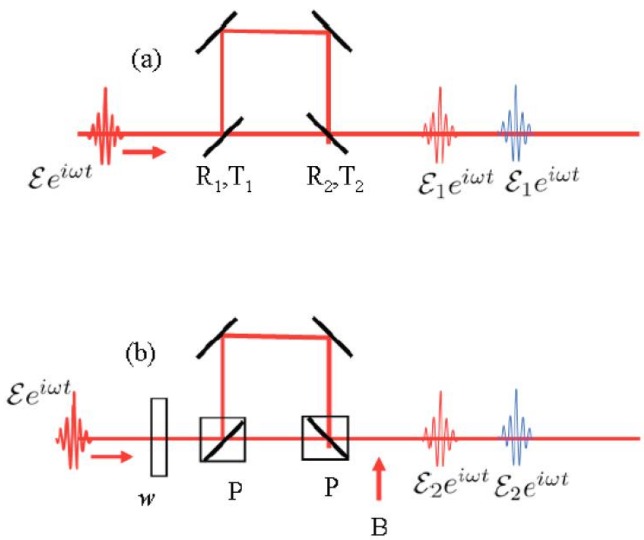
10-It is often necessary to split an electromagnetic pulse (with an electric field amplitude E) into two pulse sequence. Possible arrangements are sketched in the following figures. The input pulse is linearly polarized, and we want the two emergent pulses to be non-overlapping and also linearly polarized (parallel to each other). In both cases, the goal is to have two pulses of equal amplitude E_1 of electric field (or equal intensity)?

a) For the configuration shown in part-a, what are the value of R_1, T_1, R_2, T_2 that gives the largest E_1? For these values, what is the value of E_1 as a function of E?

b) Complete the configuration in part-b of the figure by defining two unknown elements (***w*** and **B)** to achieve the same goal (equal intensity), and find the value of E_2 as a function of E. (the two components marked as "P" are polarization beam splitters)

c) Can you explain the ratio E_1/E_2?

(15 points)



Instructions:

Answer all problems

Begin each question on a new sheet of paper!

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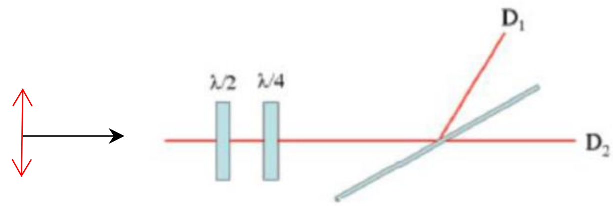
1. A linearly polarized beam is incident from the left, and passes successively through a half wave plate, a quarter wave plate and a Brewster plate. The polarization of the beam incident from the left is in the plane of the figure.



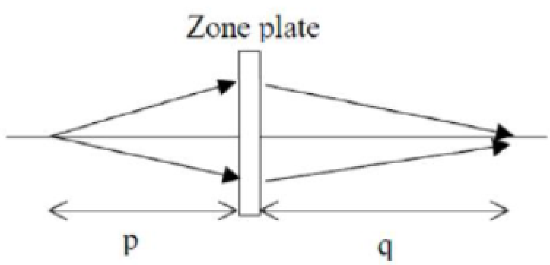
The quarter wave plate has its fast axis in the plane of the figure. The half wave plate is free to rotate around the axis of the beam.

a) Describe the state of polarization of the light as a function of the angle θ of the half wave plate (θ = 0 corresponds to the fast axis of the half wave plate in the plane of the figure)

b) Given that the index of refraction of the Brewster plate is 1.5, calculate the ratio of the reflected to transmitted intensity (D_1/D_2) as a function of the angle θ (ignore multiple reflections in the Brewster plate).

(25 points)

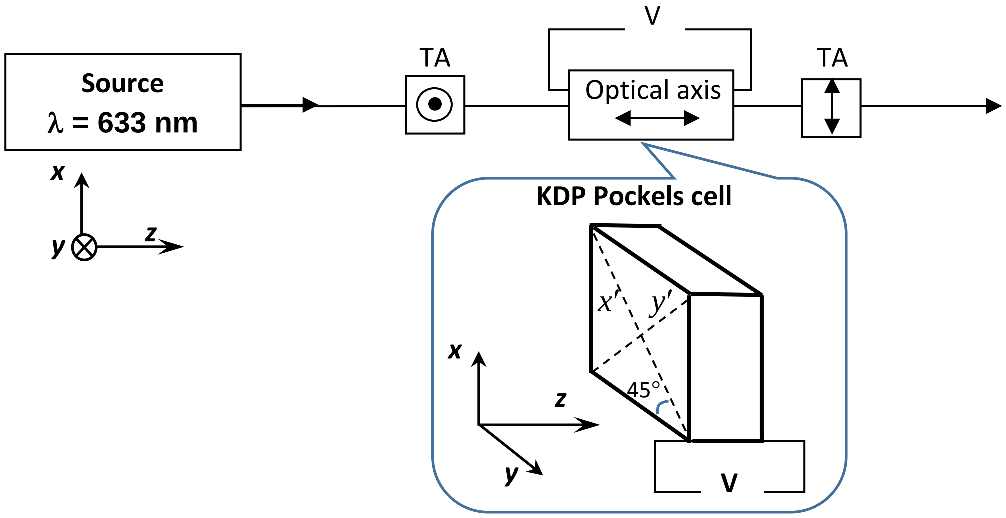
2. a) Show that in a zone plate designed for focusing a spherical wave emerging from an axial point source, the n^{th} Fresnel zone radii should be equal to $(nL\lambda)^{1/2}$ where $1/L=1/p+1/q$. (p and q are the distances of the axial source and detection point from the plate respectively).



b) A Fresnel zone plate is designed to focus the light emitted from an axial monochromatic point source 20 cm away from the plate on the opposite side. If the same plate is illuminated by a collimated beam (same wavelength), calculate the distance between the plate and the focal point with maximum intensity.

(25 points)

3-In the following configuration the source generates unpolarized light with λ= 633 nm. The transmission axis (TA) of the first polarizer is along the *y*-axis, vertical (perpendicular to the page) and TA of the second one is parallel to *x*-axis.



a) The Pockels cell is made of a 2-millimeter thick KDP slab with optical axis parallel to the incident beam. In the presence of the electric-field along the optical axis fast and slow axes (x' and y') are induced normal to z-axis with 45 degree angle relative to the original x-y axes where $n_x' \approx n_o - n_o^3 r_{63} E_z/2$ and $n_y' \approx n_o + n_o^3 r_{63} E_z/2$. KDP has ordinary and extraordinary refractive indices of n_o = 1.51 and n_e = 1.47 and linear electro-optic coefficient (r_{63}) of 10.6 pm/V. What is the half-wave voltage (V_{HW} or V_π) of the Pockel's cell?

b) Assuming the intensity of the beam incident on the first polarizer is I_0, derive a relation between the intensity of the output beam and applied voltage on the KDP crystal (clearly show the derivation steps).

(25 points)

4-A laser beam with a nominal wavelength of 1 micron and power of 1 mW is incident on a scanning Fabry-Perot. The initial distance between the two mirrors (that are identical) is such that the transmitted optical power is 1 mW.

a) If the output power is reduced to 0.5 mW by moving one of the mirrors by 1 nm, what is the reflectance of each mirror?

b) What is the quality factor of the interferometer?

c) If the laser beam has two components 100 nm apart, how long should we make the initial length in order to resolve them?

(25 points)

Instructions:

Answer all problems

Begin each question on a new sheet of paper!

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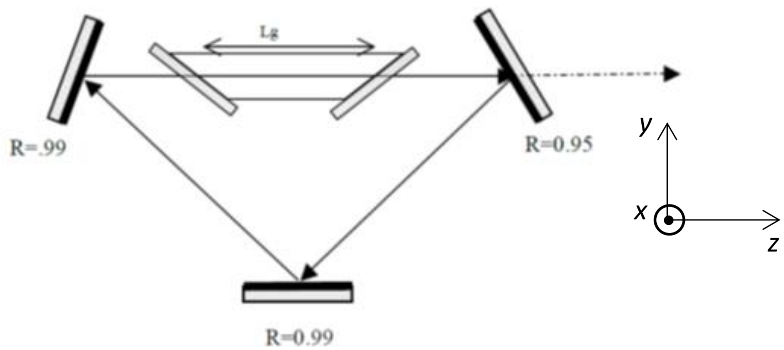
1-A unidirectional ring-cavity gas laser (with homogeneously broadened gain) is pumped 5 times above threshold to produce a cw output power of 10 Watts (the beam cross-sectional area inside the gain medium is 1 mm^2).

a) What is the saturation intensity?

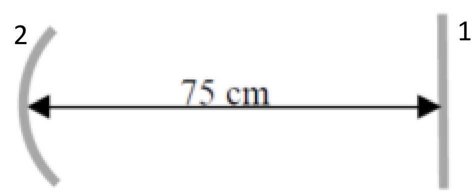
b) What is the length integrated threshold gain ($g_{th} = \gamma_{th} L_g$)?

c) If the windows (interfaces) of the gain medium are placed at the Brewster angle, what is the polarization of the output beam with respect to xyz axis (shown in the figure)?



(34 points)

2-A CO_2 laser operating at λ=10.6 μm uses a linear cavity made of a flat mirror and a spherical mirror with radius R_2=5 m (concave), as shown below:



a) Assuming $n_{CO2}\approx 1$, find the beam spot size on the spherical mirror. How far from the flat mirror (output coupler) will the diameter of the output beam be two times larger than its value on the mirror?

b) The threshold pump (P_{th}) for this laser is measured using two different output couplers with transmittances T_1 = 10% and T_1' = 5%. The other cavity mirror has reflectivity of 100%. The measured threshold pump powers are P_{th}=1 W and P_{th}'=600 mW respectively. Calculate the roundtrip internal cavity loss.

(33 points)

3-Consider the case of a transition that is broadened such that the magnitude of the gain peak $\gamma(\nu_0)$ is about 5 times larger than that of the total intracavity loss (that is almost frequency independent within the gain bandwidth). Assume that only TEM_{00} modes are allowed to oscillate, the longitudinal mode separation is only 0.3 times the FWHM (full-width half maximum) of the transition linewidth and one of these longitudinal modes coincides in frequency with the gain peak:

a) Assuming a ***Homogeneously*** broadened line: Plot the gain and loss spectra for the unsaturated case, and then add the gain spectrum for the saturated case to this plot. Discuss the lineshapes and saturation behavior.

b) Assuming a ***Inhomogeneously*** broadened line: Plot the gain and loss spectra for unsaturated case, and then add the gain spectrum for the saturated case to this plot, assuming the homogeneous linewidth for the gain atoms in this system is equal to 10% of the inhomogeneous linewidth. Discuss the lineshapes and saturation behavior.

Keep your answers short and concise.

(33 points)

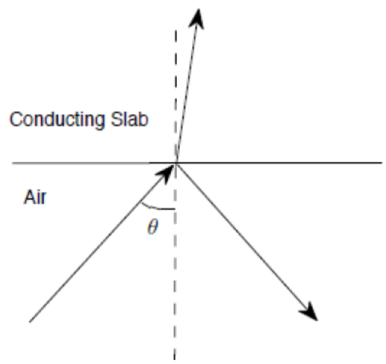
Instructions:

Solve any 3 of the 6 problems in the exam.
All problems carry equal points.

Begin each question on a new sheet of paper! and staple all pages together.

Put your Banner ID on each page.

1-A plane electromagnetic wave of angular frequency ω is incident at angle θ with respect to the normal of the surface if a semi-infinite slab occupying the region $z \geq 0$, as shown in the figure. Assume that the slab is highly conducting but non-magnetic, with a dielectric permittivity ε and conductivity σ that is large but not infinite.



a) Show from Maxwell-Ampere's law that the slab may be regarded as a dielectric with a complex effective permittivity equal to $\varepsilon + i\sigma/\omega \approx i\sigma/\omega$.

b) Show that the wave transmitted into the conductor is an evanescent plane wave. What is the characteristic depth, in terms of σ, ω and certain electromagnetic constants, to which the transmitted field propagates inside the slab in the high-conductivity limit, $\sigma >> \omega\varepsilon$?

c) Show that in the high conductivity limit, the transmitted wave propagates essentially normally to the surface, regardless of the angle of incidence. Express n in terms of σ, ω and electromagnetic constants.

2. Two electromagnetic waves, both at frequency ω and both propagating in air along the z-axis, have identical fields at $z = 0$ and $t = 0$, but one is linearly polarized along the x-axis, while the order is RCP (right circularly polarized):

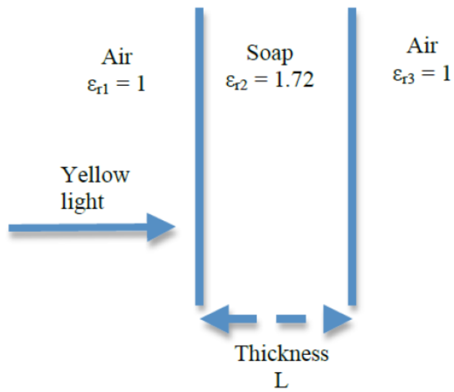
a) Express the electric and magnetic fields of the superposition of these two waves at any z and t in complex form and also as real, physical field (i.e., in terms of *Sin* and *Cos*).

b) How is the total, time-averaged energy density $\langle w \rangle$ of the superposition of these two waves related to that of the linearly polarized wave alone (find the ratio)?

3. Consider a thin film of soap in air under illumination by yellow light with λ=0.6 μm in vacuum.

a) If the film is treated as a planar dielectric slab with $\varepsilon_r = 1.72$, surrounded on both sides by air, what *minimum* film thickness would produce the strongest reflection of the yellow light at normal incidence? How much is reflected.

b) At what intervals of film thickness greater than the minimum found in part (a) is this strong reflection reproduced?



4. A left-hand circularly polarized (LHCP) wave is normally incident onto an infinite PEC surface located at $z = 0$.

a) What is the polarization and parity (sense of rotation) of the reflected wave electric field (explain your answer)? **[2]**

b) What is the normalized (maximum unity) output voltage if the reflected wave is received by a LHCP antenna? **[2]**

c) What is the normalized (maximum unity) output voltage if the reflected wave is received by a right-hand circularly polarized (RHCP) antenna? **[2]**

d) Repeat a)-c) if the PEC surface is replaced with a semi-infinite plane of water for a 10 GHz incident wave ($\varepsilon_r = 81, \sigma = 4$ S/m). **[4]**

5. A uniform plane wave is incident on a planar boundary at an incident angle θ from a dielectric medium to a half space as shown in Figure 1, where $\epsilon > \epsilon_0$.

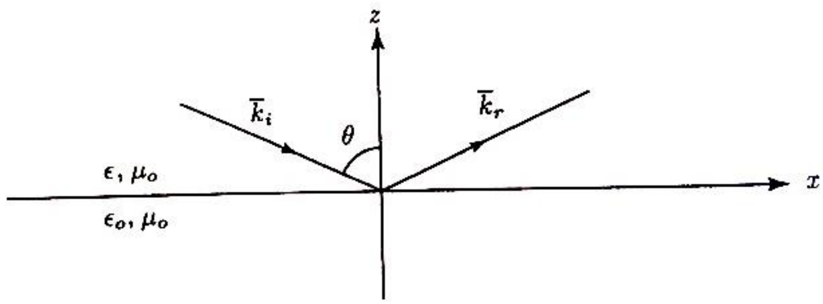


Figure 1 Configuration of the problem.

The electric field vector of the incident wave is given by

$$\mathbf{E}_i = (\cos\theta\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} + \sin\theta\hat{\boldsymbol{z}})E_0\cos(k_x x - k_z z - \omega t)$$

where the amplitude E_0 is a constant.

a) Show that this incident electric field satisfies Gauss's law. **[2]**
b) Decompose the incident electric field into its TE and TM components. **[2]**
c) Determine the expression for the transmitted electric and magnetic fields. **[2]**
d) What is the surface impedance along the planar boundary? (defined as the ratio of the tangential electric to magnetic field) **[4]**

6. The instantaneous electric field inside a conducting rectangular, free-space-filled waveguide of width a and height b is given by:

$$\tilde{\mathbf{E}} = \hat{\boldsymbol{x}} E_0 \sin\left(\frac{\pi}{a}x\right)\cos(\omega t - \beta_z z)$$

where β_z is the waveguide's phase constant. Assuming there are no sources within the waveguide. Determine the

a) Corresponding instantaneous magnetic field components inside the waveguide. **[2]**
b) Phase constant β_z. **[2]**
c) The expression for the time-harmonic electric and magnetic fields. **[3]**
d) The instantaneous Poynting vector and the time-average Poynting vector (average power density) **[3]**